AN INDEX FORMULA ON MANIFOLDS WITH FIBERED CUSP ENDS

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ABSTRACT. We consider a compact manifold X whose boundary is a locally trivial fiber bundle, and an associated pseudodifferential algebra that models fibered cusps at infinity. Using trace-like functionals that generate the 0-dimensional Hochschild cohomology groups we first express the index of a fully elliptic fibered cusp operator as the sum of a local contribution from the interior of X and a term that comes from the boundary. This leads to an abstract answer to the index problem formulated in [11]. We give a more precise answer for first-order differential operators when the base of the boundary fiber bundle is S^1. In particular, for Dirac operators associated to a metric of the form $g^X = \frac{dx^2}{x^4} + \frac{d\theta^2}{x^2} + g^F$ near $\partial X = \{x = 0\}$ with twisting bundle T we obtain

$$\operatorname{index}(A) = \int_X \hat{A}(X) \operatorname{ch}(T) - \frac{\lim_a \eta(A_{|\partial X})}{2}$$

in terms of the integral of the Atiyah-Singer form in the interior of X, and the adiabatic limit of the η-invariant of the restriction of the operator to the boundary.

1. INTRODUCTION

Let X be a compact manifold whose boundary is the total space of a locally trivial fiber bundle $\varphi : \partial X \to Y$ of closed manifolds. Let $x : X \to \overline{\mathbb{R}}_+$ be a defining function for ∂X, i.e., $\partial X = \{x = 0\}$ and dx does not vanish at ∂X. As shown by Melrose [13], the choice of an appropriate Lie algebra of vector fields (boundary fibration structure) is the first step towards a pseudodifferential calculus on X. This choice is by no means unique, and different such structures on X require in fact completely different analytic tools – see, for instance, [6, 7, 9, 10, 11, 15].

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In this paper we study the boundary fibration structure determined by the Lie algebra $\mathcal{V}_\Phi(X)$ of *fibered-cusp* or briefly Φ*-vector fields* where a smooth vector field V on X belongs to $\mathcal{V}_\Phi(X)$ provided V is tangent to the fibers of φ at the boundary and satisfies $Vx \in x^2\mathcal{C}^\infty(X)$. It is instructive to picture Φ-vector fields in local coordinates. Let y be a local coordinate system on Y lifted to ∂X via φ. Complete y to a local coordinate system (y, z) on ∂X and extend this smoothly to an open set $U \subset X$. Then $(x, y, z) : X \supseteq U \to \overline{\mathbb{R}}_+ \times \mathbb{R}^n_y \times \mathbb{R}^m_z$ are local coordinates near ∂X. In these variables, $V \in \mathcal{V}_\Phi(X)$ can be written as

$$V|_U(x, y, z) = a(x, y, z)x^2\partial_x + \sum_{j=1}^{n} b_j(x, y, z)x\partial_{y_j} + \sum_{k=1}^{m} c_k(x, y, z)\partial_z$$

with coefficients a, b_j, and c_k smooth down to $x = 0$. Taking these coefficients as local trivializations we see that the Lie algebra $\mathcal{V}_\Phi(X)$ can be realized as the space of smooth sections of a smooth vector bundle ${}^\Phi TX \to X$ that comes equipped with a natural homomorphism ${}^\Phi TX \to TX$.

The algebra of Φ-differential operators is by definition the enveloping algebra $\mathrm{Diff}^*_\Phi(X)$ of $\mathcal{V}_\Phi(X)$ over $\mathcal{C}^\infty(X)$. A corresponding Φ-pseudodifferential calculus $\Psi^*_\Phi(X)$ has been constructed by Mazzeo and Melrose in [11]. First, Φ-pseudodifferential operators act canonically on $\mathcal{C}^\infty(X)$, but since $\mathrm{Diff}^*_\Phi(X)$ as well as $\Psi^*_\Phi(X)$ are $\mathcal{C}^\infty(X)$-modules we can consider Φ-(pseudo)differential operators acting between sections of smooth vector bundles $\mathcal{E}, \mathcal{F} \to X$ over X and we write $\mathrm{Diff}^*_\Phi(X; \mathcal{E}, \mathcal{F})$ (resp. $\Psi^*_\Phi(X; \mathcal{E}, \mathcal{F})$) for the corresponding spaces. Important examples of Φ-differential operators are the Laplacian and the Dirac operators corresponding to *exact* Φ*-metrics*, for instance (1) (see [11]; these are certain complete metrics on X° which induce Euclidean metrics on ${}^\Phi TX$ over X).

Note that almost the same calculus of pseudodifferential operators can be constructed by integrating an appropriate Lie algebroid as in [19].

As in the closed case, a Φ-pseudodifferential operator of order m_0 acts continuously as an operator of order m_0 on a scale of Φ*-Sobolev spaces* H^s_Φ, $s \in \mathbb{R}$. The Φ-pseudodifferential operators that induce Fredholm operators $H^s_\Phi \to H^{s-p}_\Phi$ have been characterized by Mazzeo and Melrose [11] as being those operators with invertible principal symbol as well as invertible *normal operator* (see Section 3). Such operators are called fully elliptic. The index of a fully elliptic operator A is independent of the particular $s \in \mathbb{R}$. A preliminary index formula for fully elliptic operators has been obtained in [6] under the assumption that $\mathcal{E} =$

$\mathcal{F}$ (this assumption is not a restriction if we already have an elliptic operator from $\mathcal{E}$ to $\mathcal{F}$; an isomorphism $\mathcal{E} \to \mathcal{F}$ is given by the principal symbol of the operator applied to a non-vanishing Φ-vector field, which exists whenever $\partial X \neq \emptyset$).

Let us recall the index formula. We need several trace-like functionals on the Φ-calculus whose definition has been adapted from a similar context in [16]. Let $Q \in \Psi^1_\Phi(X)$ be a positive self-adjoint fully elliptic operator and $Q^{-\lambda} \in \Psi^{-\lambda}_\Phi(X)$ the family of complex powers constructed as in [5]. Then for $A \in \Psi^{m_0}_\Phi(X)$, the operator $x^z A Q^{-\lambda}$ is of trace class for $\Re(z) > n+1$ and $\Re(\lambda) > m_0 + \dim(X)$ (recall that $n = \dim(Y)$ and $\dim(X) = n+m+1$) and the map $(\lambda, z) \mapsto z\lambda \operatorname{Tr} x^z A Q^{-\lambda}$ admits a meromorphic extension to $\mathbb{C}^2$ with at most simple poles in each variable, which is analytic near $(z,\lambda) = (0,0)$; thus, we can define

$$\begin{aligned} z\lambda \operatorname{Tr}(x^z A Q^{-\lambda}) =: {} & \operatorname{Tr}_{\partial,\sigma}(A) + \lambda \widehat{\operatorname{Tr}}_\partial(A) + z \widehat{\operatorname{Tr}}_\sigma(A) \\ & + \lambda^2 W(\lambda, z) + \lambda z W'(\lambda, z) + z^2 W''(\lambda, z), \end{aligned}$$

where the error terms W, W', W'' are holomorphic near $0 \in \mathbb{C}^2$. The functionals obtained in this way have been studied in [6].

Theorem 1 ([6])**.** *Let $A \in \Psi^{m_0}_\Phi(M, \mathcal{E})$ be a fully elliptic Φ-operator. Then*

$$\operatorname{index} A = \widehat{\operatorname{Tr}}_\sigma(A[\log Q, B]) - \widehat{\operatorname{Tr}}_\partial([A, \log x]B)$$

where $B \in \Psi^{-p}_\Phi(X)$ is any inverse of A up to trace class remainders, and

$$[\log Q, B] := \frac{d}{d\lambda}(Q^\lambda B Q^{-\lambda})_{|\lambda=0} \in \Psi^{-p}_\Phi(X, \mathcal{E});$$

$$[\log x, A] := \frac{d}{dz}(x^z A x^{-z})_{|z=0} \in x\Psi^{p-1}_\Phi(X, \mathcal{E}).$$

This is similar to the improvement of the original computation of Melrose and Nistor [16] given in [8]. From [6] we know already that the first contribution to the index is local, i.e., does not change if we modify A by an operator of sufficiently negative order, whereas the second contribution is global but depends only on the behavior of the operators at the boundary, i.e., it does not change if we modify A by an operator that vanishes to sufficiently high order at the boundary. We identify in Proposition 16 the local term in Theorem 1 with the regularized Atiyah-Singer integral for the index, defined in terms of heat kernel asymptotics.

In general, the boundary term is rather inexplicit. In this paper we compute this boundary term (and hence the index) for a class of fully elliptic first-order Φ-differential operators which includes Dirac operators. We can treat the case where the base Y of the fiber bundle $\varphi : \partial X \to Y$ is the circle S^1. Choose a connection in φ, i.e., a rule for lifting the horizontal vector field ∂_θ. Fix a smooth metric on X° which in a product decomposition $\partial X \times [0, \varepsilon) \subset X$ close to the boundary has the form

$$g^X = \frac{dx^2}{x^4} + \frac{d\theta^2}{x^2} + g^F, \tag{1}$$

where θ is the variable in the circle and g^F is a family of metrics on the fibers. Such a metric is called a *product Φ-metric*; it induces an Euclidean metric on the vector bundle ${}^\Phi TX$.

Theorem 2. *Let X be a compact manifold with boundary fibered over S^1 and metric* (1) *over X°. Let $\mathcal{E}, \mathcal{F} \to X$ be Hermitian vector bundles together with an orthogonal decomposition $\mathcal{E}_{|\partial X} = E^+ \oplus E^-$, Hermitian connections ∇ in $E^\pm$ and an isometry $\sigma : \mathcal{E}_{|\partial X} \to \mathcal{F}_{|\partial X}$. Let*

$$A : \mathcal{C}^\infty_c(X^\circ, \mathcal{E}) \to \mathcal{C}^\infty_c(X^\circ, \mathcal{F})$$

be an elliptic first-order differential operator which in the fixed product decomposition $\partial X \times [0, 1] \subset X$ looks like

$$A = \sigma \left((x^2 \partial_x - \frac{x}{2}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -ix\tilde{\nabla}_{\partial_\theta} & D^* \\ D & ix\tilde{\nabla}_{\partial_\theta} \end{bmatrix} \right) \tag{2}$$

where D is a family of invertible operators on the fibers of φ, D^* is the formal adjoint of D, and $\tilde{\nabla}_{\partial_\theta} := \nabla_{\partial_\theta} + \frac{1}{4} \operatorname{Tr}(L_{\partial_\theta} g^F)$. Then A is Fredholm as an unbounded operator in $L^2(X, \mathcal{E}, \mathcal{F}, g^X)$, and its index is given by

$$\operatorname{index}(A) = \overline{AS}(A) - \frac{\lim_a \eta(\delta_x)}{2}.$$

Here $\overline{AS}(A)$ is the integral on X of the pointwise supertrace of the heat kernel of A, which is a local expression in the full symbol of A and in the metric, while $\lim_a \eta(\delta_x)$ is the adiabatic limit (the limit as x tends to 0) of the eta invariant of the one-parameter family of "boundary operators" indexed by $x \in [0, \infty)$

$$\delta_x := \begin{bmatrix} -ix\tilde{\nabla}_{\partial_\theta} & D^* \\ D & ix\tilde{\nabla}_{\partial_\theta} \end{bmatrix} : C^\infty(\partial X, \mathcal{E}) \to C^\infty(\partial X, \mathcal{E}). \tag{3}$$

Note that we do not need to actually construct the heat kernel of A in order to define the local index density. This density is slightly less than L^1, see Proposition 23. It is compelling to look for a proof by a passage to the limit in the classical Atiyah-Patodi-Singer formula, however this is too hard even for product fibrations.

Corollary 3. *Let (X, g^X) be a compact spin manifold with boundary and $T \to X$ a Hermitian vector bundle with constant metric and connection near ∂X. Then the twisted Dirac operator A on (X, g^X) is of the form* (2)*. Moreover, A is Fredholm on $L^2(X, \mathcal{E}, \mathcal{F}, g^X)$ if and only if the family D of Dirac operators on the fibers is invertible, and*

$$\mathrm{index}(A) = \int_X \hat{A}(X)\,\mathrm{ch}(T) - \frac{\lim_a \eta(\delta_x)}{2}.$$

This follows immediately from the local index theorem (see [2]) and Theorem 2. One checks directly in this case that the curvature of g^X is a smooth 2-form on X with values in $\mathrm{End}({}^{\Phi}TX)$ so $\hat{A}(X)\,\mathrm{ch}(T)$ is a smooth form on X, thus in L^1 (see Proposition 17).

Paolo Piazza suggested to us that Corollary 3 may be equivalent to the same statement for the conformally equivalent metric $g_d := x^2 g^X$, which is a particular case of Vaillant's index formula [21]. We stress however that Theorem 2 is inaccessible with the standard heat operator techniques. It is remarkable that while the local index formula fails for our more general operators, the boundary correction term is still the same as in the case of compatible Dirac operators.

Corollary 4. *Let A be as in Theorem 2. Then the integral of the index density is an integer if and only if the determinant bundle of the boundary family D has trivial holonomy.*

Indeed, the adiabatic limit of the eta invariant equals the logarithm of the holonomy (see Section 2 for the definitions). In particular, under the assumptions of Corollary 3 the Atiyah-Singer volume form defines an integral cohomology class.

A related index formula has been obtained by Nye and Singer [20] for the spin Dirac operator on $X = S^1 \times \mathbb{R}^3$ where the boundary fiber bundle is the projection $S^1 \times S^2 \to S^2$. More generally, Vaillant [21] gave a formula for the index of a Dirac operator of a d-metric on arbitrary manifolds with fibered cusps under some mild conditions. Vaillant's formula contains the adiabatic limit of the eta invariant in the form computed by Bismut and Cheeger [3]. It seems therefore likely that our Theorem 2 also continues to hold for boundary fiber bundles with higher dimensional base.

The paper is structured as follows: in Section 2 we recall results about the eta invariant of self-adjoint operators. Section 3 is devoted to an introduction to fibered cusp pseudodifferential operators, with a focus on traces. The index theorem 1 is reviewed in Section 4. Finally the proof of Theorem 2 occupies Section 5. A surprising feature of the proof is the appearance and then cancellation of the integral over S^1 of the determinant of D^*D in the boundary term of the index formula. As noted above, our results and Vaillant's have a non-empty intersection (Corollary 3) and we close the paper by explaining this link.

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2. Review of eta invariants and adiabatic limits

The eta function of an elliptic self-adjoint differential operator δ was defined by Atiyah, Patodi and Singer [1] as

$$\eta(\delta, s) := \mathrm{Tr}\left((\delta^2)^{-\frac{s+1}{2}}\delta\right).$$

(assuming that δ is invertible).

Lemma 5. *The operator δ_x defined in (3) for $0 \leq x < \infty$ is symmetric on $L^2(\partial X, \mathcal{E}, \frac{d\theta^2}{x^2} + g^F)$.*

Proof. We must show that $\tilde{\nabla}_{\partial_\theta}$ is skew-symmetric. Let $s_1, s_2 \in \mathcal{C}^\infty(\partial X, \mathcal{E})$. Then

$$\begin{aligned}
&(\tilde{\nabla}_{\partial_\theta} s_1, s_2) + (s_1, \tilde{\nabla}_{\partial_\theta} s_2)\\
&= \int_{\partial X} \left((\tilde{\nabla}_{\partial_\theta} s_1, s_2) + (s_1, \tilde{\nabla}_{\partial_\theta} s_2)\right) dg^F d\theta\\
&= \int_{\partial X} (\partial_\theta (s_1, s_2) + \frac{(s_1, s_2)}{2} \mathrm{Tr}(L_{\partial_\theta} g^F)) dg^F d\theta\\
&= \int_{\partial X} L_{\partial_\theta}((s_1, s_2) dg^F) d\theta\\
&= \int_{S^1} \partial_\theta \left(\int_{\partial X / S^1} (s_1, s_2) dg^F\right) d\theta = 0.
\end{aligned}$$

□

We will see in Section 5 that δ_x is also invertible for small enough $x > 0$.

The eta invariant of δ_x is by definition the regularized value of $\eta(\delta_x, s)$ at $s = 0$. In fact, the eta function is regular at $s = 0$ (see [1]). By the adiabatic limit, denoted $\lim_a \eta(\delta_x)$, we mean the limit $\lim_{x\to 0} \eta(\delta_x)$.

Intuitively it corresponds to separating the fibers of $\partial X \to S^1$ in the limit since the Riemannian distance between distinct fibers tends to infinity.

Recall the definition of the determinant line bundle of the family D with the Bismut-Freed connection. Since D is invertible, $\det(D)$ is defined as the trivial bundle $\mathbb{C} \times S^1 \to S^1$ with the connection $d + \omega^{BF}(0)$, where $\omega^{BF}(0)$ is the finite value at $s = 0$ of the meromorphic family of 1-forms

$$\omega^{BF}(s) := \operatorname{Tr}\left((D^*D)^{-\frac{s}{2}} D^{-1} \tilde{\nabla}_{\partial_\theta}(D)\right).$$

Clearly, the holonomy of the Bismut-Freed connection is

$$\operatorname{hol}(\det(D), \omega^{BF}(0)) = e^{-\int_{S^1} \omega^{BF}(0)}.$$

We define the logarithm of the holonomy of $\det(D)$ as

$$\log \operatorname{hol}(\det(D)) = -\int_{S^1} \omega^{BF}(0) \in i\mathbb{R}. \tag{4}$$

Determinant bundles and eta invariants are linked by the global anomaly formula of Witten [22], initially proved in [4] for Dirac operators. For more general operators the result that we need is taken from [17, 18].

Theorem 6. *Let D be a family of first-order elliptic differential operators on the fibers of φ, and δ_x the family of operators indexed by $x \in (0, \infty)$ defined by* (3)*. The limit as $x \to 0$ (i.e., the adiabatic limit) of the eta invariant of δ_x satisfies*

$$\lim_a \eta(\delta_x) = -\frac{1}{i\pi} \log \operatorname{hol}(\det(D)).$$

3. The fibered cusp calculus of pseudo-differential operators

In this section we introduce the basic facts about the fibered cusp calculus from [6] that are used in the next sections. For a thorough treatment of the fibered cusp calculus we refer the reader to [11, 21]. We continue to use the notations from [6].

Blowing-up a submanifold N of a smooth manifold M means replacing N by the set of its real normal directions inside M, i.e., the sphere bundle of its inner-pointing normal bundle; this procedure is equally defined for manifolds with corners. The result of the blow-up is a new manifold with corners of codimension possibly higher by 1 than those of the initial manifold.

3.1. **The construction of Φ-operators.** Let

$$X^2_\Phi := [X \times X; \partial X \times \partial X, (\partial X)^2_\varphi \times \{0\}]$$

be the fibered-cusp double space, obtained by an iterated real blow-up from X^2 as follows: first blow up the corner $\partial X \times \partial X$ (at this stage we get the celebrated b-double space X^2_b). The new boundary hyperface introduced by this blow-up is diffeomorphic to $\partial X \times \partial X \times [-1, 1]$ under the following map: the class (modulo $\mathbb{R}^*_+$) of the non-zero normal vector (V_1, V_2) at (y_1, y_2) maps to $\frac{V_1(x)-V_2(x)}{V_1(x)+V_2(x)}$. Thus $\partial X \times \partial X \times \{0\}$ is a well-defined submanifold of X^2_b provided we fixed the boundary-defining function x. The second stage of the construction involves blowing-up the fiber diagonal

$$(\partial X)^2_\varphi := \partial X \times_\varphi \partial X = \{(p, q) \in \partial X \times \partial X; \varphi(p) = \varphi(q)\}$$

of the boundary fiber bundle, which by the discussion above is also a submanifold of X^2_b. The space X^2_Φ comes equipped with a canonical smooth structure as a manifold with corners of codimension at most 2, and with a smooth blow-down map

$$\beta : X^2_\Phi \to X^2$$

which extends the identity $(X^2_\Phi)^\circ = (X^2)^\circ$ of the interiors. The last face introduced by blow-up is called the Φ-front face, denoted ff_Φ. The lifted diagonal Δ_Φ is by definition the closure in X^2_Φ of the preimage under β of the interior of the diagonal in X^2.

The motivation of the construction is the fact [11, Corollary 1] that the space of Φ-differential operators described in the Introduction is canonically isomorphic to the space of distributions on X^2_Φ supported on Δ_Φ, conormal to Δ_Φ and extendible across ff_Φ, with values in the bundle $\mathcal{F} \boxtimes \mathcal{E}^* \otimes \Omega'$. Here Ω' is the pull-back through the projection on the right factor of the Φ-density bundle $\Omega({}^\Phi TX)$. Note that

$$\mathcal{C}^\infty(X, \Omega({}^\Phi TX)) = x^{-\dim(Y)-2}\mathcal{C}^\infty(X, \Omega(TX)).$$

This singularity of order $\dim(Y) + 2$ will play a great role in the rest of the paper. In fact, the main reason for assuming $Y = S^1$ in Theorem 2 is making this order of singularity relatively small. One defines then [11] $\Psi^{m_0}_\Phi(X)$ as the space of linear operators $A : \dot{C}^\infty(X, \mathcal{E}) \to \dot{C}^\infty(X, \mathcal{F})$ such that the lift κ_A of the Schwartz kernel k_A to X^2_Φ is a classical conormal distribution of order m_0 on X^2_Φ with values in $\mathcal{E}^* \boxtimes \mathcal{F} \otimes \Omega'$, vanishing rapidly to all boundary faces other than ff_Φ and extendible across ff_Φ. These operators extend to bounded operators between appropriate Φ-Sobolev spaces. The fibered cusp calculus is closed under composition [11, Theorem 2].

If φ is the identity map, fibered-cusp operators are nothing else than *scattering* operators [12]. In that case the identifier Φ for double spaces, tangent bundles etc. will be replaced by *sc*.

3.2. **The normal operator.** There exist two symbol maps on $\Psi_\Phi(X)$, both multiplicative under composition of operators. One is the usual conormal principal symbol (living on ${}^\Phi T^*X$). To describe the second symbol $\mathcal{N}$, called the *normal operator*, first assume that $Y = S^1$. In this case the interior of the Φ-front face ff_Φ is the total space of a trivial 2-dimensional real vector bundle over $(\partial X)^2_\varphi$. The two real directions correspond to the normal direction to ∂X in X, and to the normal direction to $(\partial X)^2_\varphi$ inside $\partial X \times \partial X$. Then $\mathcal{N}$ is obtained by "freezing coefficients" at ff_Φ and then Fourier–transforming in the two real directions:

$$\mathcal{N}(A) := \widehat{\kappa_A|_{\mathrm{ff}_\Phi}}.$$

Note that in the general case there are $\dim Y + 1$ suspending variables. This new symbol map $\mathcal{N}$ surjects onto the algebra $\Psi^{\mathbb{Z}}_{\mathrm{sus}({}^\Phi N^*Y)-\varphi}(\partial X)$ of families of classical pseudo-differential operators on $\{Z_p \times \mathbb{R}^2\}_{p\in S^1}$ invariant with respect to translations in $\mathbb{R}^2$ (2-*suspended operators* in the terminology of [14]) where Z_p is the fiber over $p \in S^1$ of the boundary fiber bundle $\varphi : \partial X \to S^1$. An operator A in $\Psi_\Phi(X)$ is called *elliptic* if its principal conormal symbol is pointwise invertible on the sphere bundle of ${}^\Phi T^*X$, and *fully elliptic* if, in addition, the corresponding normal operator $\mathcal{N}(A)$ consists of a family of invertible pseudodifferential operators.

Lemma 7. *Fix a boundary-defining function x inside the fibered cusp structure, and a product decomposition $\partial X \times [0, \varepsilon)$ of X near ∂X. Let P be a family of fiberwise differential operators (i.e., composition of endomorphisms and vertical vector fields). Then*

$$\begin{aligned}
\mathcal{N}(x\tilde{\nabla}_{\partial_\theta}) &= i\tau\\
\mathcal{N}(x^2\partial_x) &= i\xi\\
\mathcal{N}(P) &= P_{|x=0}
\end{aligned}$$

where τ, ξ are the variables on $\mathbb{R}^2$.

Proof. For Φ differential operators there exists an alternate definition of $\mathcal{N}$ (see [11]):

$$\begin{aligned}&\mathcal{N}(A)(\exp(i\theta),\xi,\tau)\\ &=\left(\exp\left(i\frac{\xi+\tau(\theta-\theta')}{x}\right)A\exp\left(-i\frac{\xi+\tau(\theta-\theta')}{x}\right)\right)_{\substack{x=0\\ \theta'=\theta}}.\end{aligned}$$

The lemma follows easily. □

Note that the choices of connection, horizontal lift and product decomposition are not detected by $\mathcal{N}$.

3.3. **The formal boundary symbol.** The principal symbol and the normal operator are invariantly defined. As for standard pseudodifferential operators there exists a more refined notion of *formal symbol* map, associating to an operator the Laurent series of its full symbol at the sphere at infinity inside the radial compactification $\overline{{}^{\Phi}T^*X}$ (however, this symbol depends on choices except for its first term, the principal symbol). Similarly, we associate to a Φ operator A its *formal boundary symbol* $q(A)$:

$$q:\Psi_\Phi(X)\to\Psi_{\mathrm{sus}({}^\Phi N^*Y)-\varphi}(\partial X)[[x]]\,. \tag{5}$$

To construct q, first choose a product decomposition $\partial X\times[0,\varepsilon)\hookrightarrow X$ of X near ∂X so that $x(y,t)=t$. Let X_ε be the image of this map, and $Y_\varepsilon:=Y\times[0,\varepsilon)$. Thus X_ε fibers over Y_ε via $\varphi\times Id$ with fiber type F and $(X_\varepsilon)^2_\Phi$ fibers over $(Y_\varepsilon)^2_{sc}$ with fiber type $F\times F$ [6, Section 3].

Lift through β the diagonal embedding $(\partial X)^2_\varphi\times[0,\varepsilon)\hookrightarrow X^2$ to an embedding

$$(\partial X)^2_\varphi\times[0,\varepsilon)\hookrightarrow X^2_\Phi. \tag{6}$$

Note that $(\partial X)^2_\varphi\times\{0\}$ maps identically to itself as the zero section in ff_Φ. Moreover, the image of (6) is exactly the preimage of Δ_{sc} under the fibration

$$(X_\varepsilon)^2_\Phi\to(Y_\varepsilon)^2_{sc}. \tag{7}$$

Thus the normal bundle to $(\partial X)^2_\varphi\times[0,\varepsilon)$ inside X^2_Φ is the pull-back via (7) of the normal bundle to Δ_{sc}, which is canonically isomorphic to ${}^{sc}TY_\varepsilon$. Consequently we use the notation

$$N((\partial X)^2_\varphi\times[0,\varepsilon))=(\partial X)^2_\varphi\times_Y{}^{sc}TY_\varepsilon=:{}^{sc}TX_\varepsilon.$$

The total space of ${}^{sc}TX_\varepsilon|_{x=0}$ coincides with the interior of the front face, while the zero section of ${}^{sc}TX_\varepsilon$ is included in X^2_Φ by Eq. (6). Choose an open neighborhood $\mathcal{U} \subset {}^{sc}TX_\varepsilon$ of ${}^{sc}TX_\varepsilon|_{x=0} \bigcup (\partial X)^2_\varphi \times [0,\varepsilon)$, a collar neighborhood map $\mu : {}^{sc}TX_\varepsilon \supset \mathcal{U} \hookrightarrow X^2_\Phi$ and a cut-off function χ equal to 1 on ${}^{sc}TX_\varepsilon|_{x=0} \bigcup (\partial X)^2_\varphi \times [0,\varepsilon)$ and supported in $\mathcal{U}$.

Given $A \in \Psi_\Phi(X, \mathcal{E}, \mathcal{F})$ let κ_A be its lifted Schwartz kernel. Then $\chi\mu^*(\kappa_A)$ is a well-defined distribution on ${}^{sc}TX_\varepsilon$. Choose covariant derivatives in $\mathcal{E}, \mathcal{F}$ and trivialize $\mathcal{E} \boxtimes \mathcal{F}$ radially in the fibers of ${}^{sc}TX_\varepsilon$. Note that $\mu^*(\Omega') = \Omega_{fiber,R} \otimes \pi^*(\Omega({}^{sc}TY_\varepsilon))$ is the tensor product of the density bundle in the right factor of $(\partial X)^2_\varphi$ and the pull-back of the scattering density bundle from Y_ε. This last factor allows us to take the Fourier transform of $\chi\mu^*(\kappa_A)$ in the fibers. Recall that besides the tautological map ${}^{sc}TY_\varepsilon \to TY_\varepsilon$, there exists an isomorphism

$$ {}^{sc}TY_\varepsilon \to TY_\varepsilon \qquad x^2\partial_x \mapsto \partial_x, \qquad x\partial_{y_i} \mapsto \partial_{y_i}. \tag{8} $$

It follows that ${}^{sc}TX_\varepsilon$ is also a product:

$$ {}^{sc}TX_\varepsilon \simeq (\partial X)^2_\varphi \times_Y TY \times T[0,\varepsilon). $$

Therefore $\tilde{q}(A,x) := \widehat{\chi\mu^*(\kappa_A)}$ is a smooth family of suspended operators in $\Psi_{\mathrm{sus}({}^\Phi N^*Y)-\varphi}(\partial X)$ indexed by $x \in [0,\varepsilon)$. The Taylor series of $\tilde{q}(A,x)$ at $x = 0$ is by definition $q(A)$. Note that, unlike $\tilde{q}(A,x)$, $q(A)$ is independent of the choice of χ.

We now specialize to the case $Y = S^1$, so TY_ε is the trivial bundle $\mathbb{R}^2$ over $S^1 \times [0,\varepsilon)$. Let $(\tau,\xi) \in \mathbb{R}^2$ be the cotangent variables in ${}^{sc}T^*Y_\varepsilon$ in the trivialization induced by the dual of (8). Fix a connection in φ. For $u \in \mathbb{R}$ denote by ρ_u the horizontal lift to ∂X of the rotation by angle u. Then we define μ for $x > 0$, $(y_1, y_2) \in (\partial X)^2_\varphi$ and $u, v \in \mathbb{R}$ by

$$ \mu(y_1, y_2, ux\partial_\theta, vx^2\partial_x, x) := ((y_1, x), (\rho_{xu}(y_2), x + x^2 v)). $$

with values in the interior of X^2_Φ. This glues smoothly at $x = 0$ with the canonical identification of ${}^{sc}TX_{\varepsilon|x=0}$ with the interior of ff_Φ. Fix a product connection in $\mathcal{E}$ over $\partial X \times [0,\varepsilon)$ and identify $\mathcal{E}_{\rho_{xu}(y_1)}$ with $\mathcal{E}_{y_1}$ using parallel transport along the lifted rotation curve.

Proposition 8. *Assume that the product decomposition, connection in φ and covariant derivative in $\mathcal{E}$ used in the definition of μ are those from Theorem 2. Then the resulting map q satisfies*

$$ \begin{aligned} q(x\tilde{\nabla}_{\partial_\theta}) &= i\tau \\ q(x^2\partial_x) &= i\xi \\ q(P) &= [P] \end{aligned} $$

and more generally

$$q\left(x^a P(x\tilde{\nabla}_{\partial_\theta})^\alpha x^{2\beta}\partial_x^\beta\right)(\xi,\tau) = i^{\alpha+\beta}x^a[P]\tau^\alpha\xi^\beta,$$

for any family P of vertical differential operators depending smoothly on x, where $[P]$ denotes the Taylor series of P at $x=0$.

Proof. Note that the leading part of q is already fixed by Lemma 7.

For $x>0$ the map μ may as well be considered as being defined on $(\partial X)^2_\varphi \times \mathbb{R}^2 \times [0,\infty)$, since then ${}^{sc}TY_\varepsilon$ and TY_ε are equal. The fibration with connection φ is locally a product with the fiber F (since the base is 1-dimensional) so near the zero-section $\{u=0, v=0\}$, the map μ takes the following form: for $p\in S^1$, $z_1, z_2\in F$,

$$\mu(p, z_1, z_2, u\partial_x, v\partial_\theta, x) = ((p, z_1, x), (\rho_u(p), z_2, x+v)).$$

The pull-back of the Schwartz kernels of the operator $x^a P\tilde{\nabla}^\alpha_{\partial_\theta}\partial_x^\beta$ under this map is $\delta(z_1,z_2)\delta(u)\delta(v)x^a P_{z_2}\partial_u^\alpha\partial_v^\beta$ (we have used the fact that we trivialize $\mathcal{E}$ using precisely $\tilde{\nabla}$). It is superfluous to cut off since the support is already inside the zero section. By Fourier transform in (u,v) we get

$$\mathcal{F}(\mu^*(x^a P\tilde{\nabla}^\alpha_{\partial_\theta}\partial_x^\beta))(\Xi, T) = x^a P i^{\alpha+\beta} T^\alpha \Xi^\beta$$

as a family of operators on F indexed by T^*Y_ε. We now push forward this operator-valued symbol to $(\partial X)^2_\varphi \times_Y {}^{sc}T^*Y_\varepsilon$ via the tautological map (this amounts to doing nothing for $x>0$) and use the dual of (8) to trivialize ${}^{sc}T^*Y_\varepsilon$, which amounts to the change of variables

$$\Xi = \xi/x^2, \qquad T = \tau/x. \tag{9}$$

Therefore $\tilde{q}(x^a P\tilde{\nabla}^\alpha_{\partial_\theta}\partial_x^\beta))(\xi,\tau) = x^{a-\alpha-2\beta}Pi^{\alpha+\beta}\tau^\alpha\xi^\beta$, and the result follows by taking Taylor series at $x=0$. □

3.4. **Product on formal boundary symbols.** The formal boundary map q depends on choices of connections and trivializations except for its leading term which is just the normal operator. Recall that in the standard pseudo-differential case a right quantization as above induces a formal symbol map so that the induced product on formal series of homogeneous symbols (the so-called star product) takes the form

$$a(y,\xi) * b(y,\xi) = a(y,\xi)b(y,\xi) + \frac{1}{i}\sum \partial_{\xi_j}a(y,\xi)\nabla_{\partial_{y_j}}b(y,\xi) + \dots$$

Similarly, identify $\mathrm{Diff}(F\times\mathbb{R}^k, \mathcal{E})$ with the algebra of families of differential operators on F with polynomial coefficients in the fibers of $T^*\mathbb{R}^k$

via the right quantization map. As above, we fix a covariant derivative ∇ to trivialize $\mathcal{E}$ on $\mathbb{R}^k$ before taking Fourier transforms.

Lemma 9. *Assume that the curvature of ∇ vanishes on $\{z\} \times \mathbb{R}^k$ for each $z \in F$. Then the product on $\mathrm{Diff}(F \times \mathbb{R}^k, \mathcal{E})$ takes the form*

$$A(y,\Upsilon) * B(y,\Upsilon) = \sum_{J\in\mathbb{N}^k} \frac{1}{i^{|J|}J!} \frac{\partial^J A}{\partial^J_\Upsilon} \nabla^J_{\partial_y} B. \tag{10}$$

Proof. Note first that the right-hand side of (10) has a finite number of terms for each fixed A, B; moreover the order of covariant derivatives is irrelevant since the curvature vanishes. We identify $\mathcal{E}$ with its pull-back from $F \times \{0\}$ by radial parallel transport. The condition on the curvature means that $\nabla_{\partial_{y_j}}$ becomes the partial derivative ∂_{y_j} under this identification. If F is reduced to a point, Eq. (10) with $\nabla = d$ is the usual Moyal product. In general, $\mathrm{Diff}(F \times \mathbb{R}^k, \mathcal{E})$ is isomorphic to $\mathrm{Diff}(F, \mathcal{E}) \otimes \mathrm{Diff}(\mathbb{R}^k)$ so Eq. (10) follows from the Moyal formula with coefficients in the algebra $\mathrm{Diff}(F, \mathcal{E})$. □

We denote by $*$ the product on $\Psi^m_{\mathrm{sus}({}^\Phi N^*Y)-\varphi}(\partial X)[[x]]$ induced by q, so by definition the map q is multiplicative.

Lemma 10. *For $U, V \in \Psi^{\mathbb{Z}}_{\mathrm{sus}({}^\Phi N^*Y)-\varphi}(\partial X)[[x]]$, the product induced by q takes the form*

$$\begin{aligned} U * V &= UV + \frac{x}{i}\frac{\partial U}{\partial \xi}\left(x\frac{\partial V}{\partial x} + \tau\frac{\partial V}{\partial \tau} + 2\xi\frac{\partial V}{\partial \xi}\right) \\ &\quad + \frac{x}{i}\frac{\partial U}{\partial \tau}\tilde{\nabla}_{\partial_\theta}(V) + O(x^2) \\ &= \mathrm{Prod}\big[(1\otimes 1 - ix\partial_\xi \otimes (x\partial_x + \tau\partial_\tau + 2\xi\partial_\xi) \\ &\quad -ix\partial_\tau \otimes \tilde{\nabla}_{\partial_\theta} + O(x^2))(U\otimes V)\big] \end{aligned} \tag{11}$$

where the product in the right-hand side is the standard product of power series with coefficients in the algebra $\Psi^{\mathbb{Z}}_{\mathrm{sus}({}^\Phi N^*Y)-\varphi}(\partial X)$, and the error term increases the total x degree by at least 2.

Proof. Note that although it involves multiplication by x^2, the term $ix\partial_\xi \otimes x\partial_x$ increases the x-degree only by 1. It is enough to prove the formula for Φ-differential operators since the product is given by bi-differential operators with polynomial coefficients along the fibers (see e.g. [9, Proposition 3.11] for details of this argument in a similar context). Furthermore, it is enough to prove the formula over $\varphi^{-1}(I) \times (0, \varepsilon)$, where I is an interval in S^1. But then we are in the setting of Lemma 9 (since we can integrate horizontal curves to get a product decomposition of φ together with its connection over I), so the product of $\tilde{q}(A)$ and $\tilde{q}(B)$ has the form (10) with $\Upsilon = (\Xi, T)$. Under the change of variables (9),

$$\partial_\Xi \mapsto x^2\partial_\xi \qquad\qquad \partial_T \mapsto x\partial_\tau$$

$$\partial_\theta \mapsto \partial_\theta \qquad\qquad \partial_x \mapsto \partial_x + \frac{2\xi}{x}\partial_\xi + \frac{\tau}{x}\partial_\tau.$$

We remark here that the terms containing $|J|$ partial derivatives from (10) increase the x degree by exactly $|J|$. Thus, the limited Taylor expansion of (10) at $x = 0$ consists of the first two terms ($|J| = 0, 1$). □

3.5. **Traces densities of Φ-operators.** Of main interest for us are traces of Φ-operators. We study them using the more refined notion of trace density. It is a standard fact that on a closed manifold M, any operator $A \in \Psi^\lambda(M, \mathcal{E})$ with $\Re(\lambda) < -\dim(M)$ is of trace class, and

$$\mathrm{Tr}(A) = \int_\Delta \mathrm{tr}(k_A|_\Delta)$$

(the Schwartz kernel k_A is continuous on $M \times M$ and its restriction to the diagonal is a smooth 1-density with values in $\mathrm{End}(\mathcal{E})$). The same remains true for Φ-operators modulo an integrability issue at the boundary. By pulling-back via the blow-down map we write

$$\mathrm{Tr}(A) = \int_{\Delta_\Phi} \mathrm{tr}(\kappa_A|_{\Delta_\Phi}).$$

We identify Δ_Φ with X via the projection on the right factor. Recall the notation $n = \dim(Y)$, $N = \dim(X)$. The restriction to $\Delta_\Phi \cong X$ of the density bundle Ω' is precisely $\Omega({}^\Phi TX) = x^{-n-2}\Omega(X)$. It is then clear that A is of trace class if and only if $A \in x^z\Psi_\Phi^\lambda(X, \mathcal{E})$ with $\Re(z) > n + 1, \Re(\lambda) < -N$. The density $\mathrm{tr}(\kappa_A|_{\Delta_\Phi})$, viewed as a density on X, is called the *trace density* of A.

Recall that for the definition of the formal boundary symbol we have chosen a product decomposition of X near ∂X, as well as a local isomorphism of $\mathcal{E}$ with its pull-back from ∂X. We can thus expand $\kappa_A|_{\Delta_\Phi}$ in powers of x near $x = 0$. Let ${}^\Phi N\partial X$ be the restriction of the vector bundle ${}^{sc}TX_\varepsilon$ to $\partial X \times \{0\}$ embedded diagonally in ff_Φ, and ${}^\Phi N^*\partial X$ its dual.

Proposition 11. *Let $A(\lambda) \in x^z\Psi_\Phi^\lambda(X, \mathcal{E})$ be a holomorphic family of Φ-operators. Then $\lambda \mapsto \kappa_{A(\lambda)}|_{\Delta_\Phi}$ extends to a meromorphic family of* *densities on Δ_Φ with the following asymptotic expansion at $x = 0$:*

$$\kappa_{A(\lambda)}|_{\Delta_\Phi} \sim_{x\to 0} \frac{1}{(2\pi)^{n+1}} \int_{{}^\Phi N^*\partial X/\partial X} \kappa_{q(A(\lambda))} \frac{\omega_{sc}^{n+1}}{(n+1)!}$$

where ω_{sc} is the canonical (singular) symplectic form on ${}^{sc}T^*Y_\varepsilon$ pulled back to ${}^{sc}T^*X_\varepsilon$.

Proof. Note that in the right-hand side, $q(A(\lambda))$ is a formal series in x. At a pole, the meaning of the expansion is the equality of the residues and of the regularized parts. First set $A := A(\lambda)$ for a fixed λ with $\Re(\lambda) < -N$. The statement is local near $x = 0$, so we can suppose that κ_A, a distribution with continuous kernel, is supported in a neighborhood of ff_Φ. By hypothesis the cut-off function χ does not affect the restriction of $\chi\mu^*\kappa_A$ to the zero section. The Fourier inversion formula in each fiber of ${}^{sc}TX_\varepsilon$ shows

$$\kappa_A|_{(\partial X)^2_\varphi\times[0,\varepsilon)} = \frac{1}{(2\pi)^{n+1}} \int_{{}^{sc}T^*X_\varepsilon/(\partial X)^2_\varphi\times[0,\varepsilon)} \kappa_{\tilde{q}(A)} \frac{\omega_{sc}^{n+1}}{(n+1)!}.$$

Further restricting this identity to $\Delta_{\partial X}\times[0,\varepsilon)$ gives

$$\kappa_{A(\lambda)}|_{\Delta_{\partial X}\times[0,\varepsilon)} = \frac{1}{(2\pi)^{n+1}} \int_{\Delta_\Phi\times_Y {}^{sc}T^*Y_\varepsilon/\Delta_\Phi} \kappa_{\tilde{q}(A(\lambda))} \frac{\omega_{sc}^{n+1}}{(n+1)!} \tag{12}$$

for $\Re(\lambda) < -N$. It is routine to show that $\kappa_{\tilde{q}(A(\lambda))}|_{\Delta_\Phi\times_Y {}^{sc}T^*Y_\varepsilon}$ and $\kappa_{A(\lambda)}|_{\Delta_\Phi}$ extend meromorphically to $\mathbb{C}$ (by pulling back $\kappa_{A(\lambda)}$ to ${}^\Phi TX$ and by the Fourier inversion formula, the statement is reduced to a meromorphic extension result for integrals of analytic families of classical symbols). By unique continuation, (12) holds on $\mathbb{C}$. Take now Taylor series at $x = 0$ in both sides to get the result (remember that ω_{sc}^{n+1} introduces a singularity of order $n+1$). □

In the case $Y = S^1$ this takes a somewhat simpler form in which the singularity is more evident.

Corollary 12. *Let $A(\lambda) \in x^z\Psi_\Phi^\lambda(X,\mathcal{E})$ be an entire family of Φ-operators and assume $Y = S^1$. Then*

$$\kappa_{A(\lambda)}|_{\Delta_\Phi} \sim_{x\to 0} \left(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \kappa_{q(A(\lambda))}(\tau,\xi)d\tau d\xi\right)\frac{d\theta dx}{x^3}.$$

Proof. Observe that ${}^\Phi N\partial X$ is a trivial $\mathbb{R}^2$ bundle in this case, while

$$\omega_{sc} = \frac{d\xi\wedge dx}{x^2} + \frac{d\tau\wedge d\theta}{x} - \tau\frac{dx\wedge d\theta}{x^2}.$$ □

We close this section with a description in terms of the map q of the trace functional $\widehat{\mathrm{Tr}}_\partial$ defined in the Introduction (compare with [6, Proposition 7.6]).

Lemma 13. *Let $A \in \Psi_\Phi^{m_0}(X)$. Then $\widehat{\mathrm{Tr}}_\partial(A)$ is explicitly given by*

$$\widehat{\mathrm{Tr}}_\partial(A) = \frac{1}{(2\pi)^2}\left(\int_{S^1\times\mathbb{R}^2} \mathrm{Tr}(q(AQ^{-\lambda})_{[-2]}) d\theta d\tau d\xi\right)_{\lambda=0}$$

where Tr *in the right-hand side denotes the trace of operators on the fibers of $\varphi : \partial X \to S^1$, $(\cdot)_{[k]}$ is the coefficient of x^{-k} in a power expansion in x, and $(\cdot)_{\lambda=0}$ stands for the regularized value at $\lambda = 0$.*

Proof. We can assume that $\kappa_{AQ^{-\lambda}}$ is supported in $(X_\varepsilon)^2_\Phi$. Then

$$\begin{aligned}
\widehat{\mathrm{Tr}}_\partial(A) &= \mathrm{Res}_{z=0} \mathrm{Tr}(x^z AQ^{-\lambda})_{|\lambda=0} \\
&= \mathrm{Res}_{z=0} \int_{\Delta_\Phi} \mathrm{tr}(\kappa_{x^z AQ^{-\lambda}})_{|\lambda=0} \\
&= \mathrm{Res}_{z=0} \int_0^\varepsilon \int_{\partial X} x^z \,\mathrm{tr}(\kappa_{AQ^{-\lambda}})_{|\lambda=0}
\end{aligned}$$

Obviously,

$$\mathrm{Res}_{z=0}\left(\int_0^\varepsilon x^{z-k} dx\right) = \begin{cases} 1 & \text{if } k = 1; \\ 0 & \text{otherwise,} \end{cases}$$

so, using Corollary 12,

$$\begin{aligned}
\widehat{\mathrm{Tr}}_\partial(A) &= \left(\int_{\partial X} \mathrm{tr}(\kappa_{AQ^{-\lambda}})_{[1]}\right)_{|\lambda=0} \\
&= \left(\int_{S^1}\int_{\partial X/S^1}\left(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \mathrm{tr}\,\kappa_{q(AQ^{-\lambda})_{[-2]}} d\tau d\xi\right) d\theta\right)_{|\lambda=0}.
\end{aligned}$$

Now

$$\int_{\partial X/S^1} \mathrm{tr}\,\kappa_{q(AQ^{-\lambda})_{[-2]}} = \mathrm{Tr}(q(AQ^{-\lambda})_{[-2]})$$

so the result follows by Fubini's theorem. □

4. The abstract index formula

The results of this section hold for general boundary fiber bundles, i.e., not necessarily with base S^1.

Fibered-cusp operators have two types of principal symbols. Accordingly, elliptic regularity has a new aspect in fibered-cusp theory concerning regularity at the boundary. The proof of the following lemma is standard; we include it for future reference.

Lemma 14. *Let $A \in \Psi_\Phi(X, \mathcal{E}, \mathcal{F})$ be fully elliptic. Then the L^2 solutions of $A\psi = 0$ belong to $x^\infty C^\infty(M, \mathcal{E})$.*

Proof. Since A is fully elliptic there exists a parametrix B of A inverting A up to $R \in x^\infty \Psi_\Phi^{-\infty}(X, \mathcal{E})$. Let ψ be a distributional solution of the pseudo-differential equation $A\psi = 0$. It follows

$$0 = BA\psi = (I + R)\psi = \psi + R\psi$$

so $\psi = -R\psi$. But $R \in x^\infty \Psi_\Phi^{-\infty}(X, \mathcal{E})$ implies $R\psi \in x^\infty C^\infty(M, \mathcal{E})$. □

Since R is compact on L^2_Φ (the compact operators in $\Psi_\Phi^{\mathbb{Z}}(X)$ are precisely those in $x\Psi_\Phi^{-1}(X)$) it follows that $\ker A$ is finite dimensional and moreover the orthogonal projection $P_{\ker A}$ belongs to the ideal $x^\infty \Psi_\Phi^{-\infty}(X, \mathcal{E})$. We can define therefore invertible Φ-operators

$$Q_1 := (AA^* + P_{\ker A^*})^{1/2},$$
$$Q_2 := (A^*A + P_{\ker A})^{1/2}.$$

Note that $q(Q_1) = q(AA^*)^{1/2}$. Let

$$B := A^* Q_1^{-2} = Q_2^{-2} A^*$$

be a parametrix of A.

4.1. **The index formula.** Let us reprove the index formula from [6]. Assume for simplicity that A is of order 1. For technical reasons we would like to work with operators acting from $\mathcal{E}$ to itself. Recall that $\partial X \neq \emptyset$ implies the existence of a nowhere-vanishing section in ${}^\Phi T^* X$ (since the dimension of the fiber of ${}^\Phi T^* X$ equals $\dim X$, the obstruction to the existence of such a section lives in $H^{\dim(X)}(X)$ and this space is 0 when $\partial X \neq \emptyset$). The principal conormal symbol of A evaluated on this section gives an isomorphism u between $\mathcal{E}$ and $\mathcal{F}$. Finally, $v := u^*(uu^*)^{-1/2}$ is an isometry $\mathcal{F} \to \mathcal{E}$. Thus $U := vA \in \Psi_\Phi^1(X, \mathcal{E})$ has the property

$$R_1 := (UU^* + P_{\ker U^*})^{1/2} = vQ_1v^*$$
$$R_2 := (U^*U + P_{\ker U})^{1/2} = Q_2.$$

Set

$$V := Bv^* = U^* R_1^{-2} = R_2^{-2} U^*.$$

Note the commutations

$$UR_2^{-\lambda} = R_1^{-\lambda} U, \qquad VR_1^{-\lambda} = R_2^{-\lambda} V.$$

The index formula is obtained as follows:

$$\begin{aligned} \operatorname{index}(A) &= \operatorname{index}(U) \\ &= \operatorname{Tr}(UV - VU) \\ &= \operatorname{Tr}(x^z(UV - VU)R_2^{-\lambda})_{\lambda=0,z=0}. \end{aligned}$$

Recall that $\operatorname{Tr}[C,D] = 0$ for $C \in x^c\Psi_\Phi^a(X)$, $D \in x^d\Psi_\Phi^b(X)$ with $a,b,c,d \in \mathbb{C}$, $\Re(a+b) < -\dim(X)$, $\Re(c+d) > 1 + \dim(Y)$. Apply this to U and $x^z VR_1^{-\lambda}$, and use $UR_2^{-\lambda} = R_1^{-\lambda}U$. The identity below holds for large real parts of λ and z, hence for any $\lambda, z \in \mathbb{C}$ by unique continuation:

$$\begin{aligned} \operatorname{Tr}(x^z(UV - VU)R_2^{-\lambda}) &= \operatorname{Tr}(x^z UVR_2^{-\lambda} - Ux^z VR_1^{-\lambda}) \\ &= \operatorname{Tr}(x^z UV(R_2^{-\lambda} - R_1^{-\lambda}) + [x^z, U]VR_1^{-\lambda}) \\ &= \operatorname{Tr}(x^z(R_2^{-\lambda} - R_1^{-\lambda}) + \operatorname{Tr}([x^z, A]BQ_1^{-\lambda}) \\ &\quad + \operatorname{Tr}(x^z(UV - 1)(R_2^{-\lambda} - R_1^{-\lambda})) \end{aligned} \tag{13}$$

since $[x^z, U]VR_1^{-\lambda} = v[x^z, A]BQ_1^{-\lambda}v^*$ and by the trace property,

$$\operatorname{Tr}([x^z, U]VR_1^{-\lambda}) = \operatorname{Tr}([x^z, A]BQ_1^{-\lambda}).$$

Also, $UV - 1 \in x^\infty\Psi_\Phi^{-\infty}(X, \mathcal{E})$ so $\operatorname{Tr}(x^z(UV-1)(R_2^{-\lambda} - R_1^{-\lambda}))$ is holomorphic on $\mathbb{C}^2$ and vanishes at $\lambda = 0$. It follows that

$$\operatorname{index}(A) = (\operatorname{Tr}(x^z(R_2^{-\lambda} - R_1^{-\lambda}))_{\lambda=0,z=0} + \operatorname{Tr}([x^z, A]BQ_1^{-\lambda})_{\lambda=0,z=0}. \tag{14}$$

For the sake of clarity we stress the main point of such Melrose-Nistor style computations: the evaluation at $\lambda = 0, z = 0$ occurs at a point where the operators involved are not of trace class; therefore, although the operators inside the trace do vanish for $\lambda = 0, z = 0$, their regularized trace may be non-zero. We mentioned in the Introduction (see also [6]) that the meromorphic extension of the trace of a holomorphic family of Φ operators may have first-order poles (in both λ and z) at the origin. Vanishing of the family e.g., at $\lambda = 0$ implies regularity of the extension at $\lambda = 0$.

For future reference, note that the left-hand side and the third term in the right-hand side of (13) are holomorphic on $\mathbb{C}^2$. The first term of the right-hand side is regular in λ at $\lambda = 0$, the second is regular in

z at $z = 0$ because the families of operators involved vanish at those values. The identity (13) implies

Corollary 15. *Both terms in the right-hand side of* (14) *are regular both in* λ *at* $\lambda = 0$ *and in* z *at* $z = 0$.

4.2. **The interior term.** We claim that the first term in (14) is the regularized integral on X of a local expression in the full symbol expansion of U. Indeed, for $j = 1, 2$ let $r_j(\lambda) \in C^\infty(X, \mathrm{End}(\mathcal{E}) \otimes {}^\Phi\Omega(X))$ be the meromorphic extension of the lifted Schwartz kernel $\kappa_{R_j^{-\lambda}}$ restricted to Δ_Φ. As in the case of closed manifolds it is easy to see that $r_j(\lambda)$ is regular at $\lambda = 0$ (the residue of the possible pole is the residue density of the identity). The first term in (14) is $\int_X x^z \,\mathrm{tr}(r_2(0) - r_1(0))|_{z=0}$. By Corollary 12 and the remark after it, the density $\mathrm{tr}(r_2(0))$ has a Laurent expansion at $x = 0$ starting with x^{-3} so the previous integral is absolutely integrable and holomorphic in z for $\Re(z) > 2$ and extends to $\mathbb{C}$ with possible simple poles at $z = 2 - \mathbb{N}$. Now

$$\begin{aligned} R_2^{-\lambda} - R_1^{-\lambda} &= U[V, R_2^{-\lambda}] + O(\lambda)x^\infty \Psi_\Phi^{-\infty}(X) \\ &= \lambda U[\log R_2, V]R_2^{-\lambda} + O(\lambda)x^\infty \Psi_\Phi^{-\infty}(X) + O(\lambda^2)\Psi_\Phi^\lambda(X) \end{aligned}$$

where $O(\lambda)$ denotes an analytic multiple of λ near $\lambda = 0$. Clearly then

$$(\mathrm{Tr}(x^z(R_2^{-\lambda} - R_1^{-\lambda}))_{\lambda=0,z=0} = \widehat{\mathrm{Tr}}_\sigma(U[\log R_2, V]). \tag{15}$$

On the more refined level of trace densities, by [6, Proposition 7.4],

$$r_2(0) - r_1(0) = \frac{1}{(2\pi)^N} \int_{{}^\Phi S^*X/X} \sigma_{[-N]}(U[\log R_2, V]) \imath_{\mathcal{R}} \omega_\Phi^N$$

is given in terms of the component of homogeneity $-\dim(X)$ of the formal symbol of $U[V, \log R_2]$, so clearly depends only on the jets of the full symbol of U.

4.3. **The boundary term.** Similarly for the second term from (14) we have

$$[x^z, A]B = zx^z[\log x, A]B + O(z^2)$$

and $\mathrm{Tr}(O(z^2)) = O(z)$ so

$$\mathrm{Tr}([x^z, A]BQ_1^{-\lambda})_{\lambda=0,z=0} = -\widehat{\mathrm{Tr}}_\partial([A, \log x]B). \tag{16}$$

By Lemma 13 this last quantity is concentrated at the boundary.

So far, combining (14), (15) and (16) we have proved the general index Theorem 1. Note that in (15) we can assume $v = 1$ (and so $U = A$, $V = B$, $Q = R_2$) since in Theorem 1 we suppose $\mathcal{E} = \mathcal{F}$.

4.4. **Relationship with heat kernel expansions.** Here is a more familiar interpretation of the local term (15). It is worth stressing that we do not prove a heat kernel expansion for A^*A. Rather we use the existence of heat kernel expansions for pseudo-differential operators on closed manifolds as well as the locality of the two quantities we want to relate.

Proposition 16. *The local quantity* $\operatorname{tr}(r_2(0) - r_1(0))$ *equals the index density, defined as the pointwise supertrace of the constant term in the heat kernel expansion of* $\left[\begin{smallmatrix} 0 & A^* \\ A & 0 \end{smallmatrix}\right]^2$.

Proof. Fix a point p in the interior of X and modify the operator U far from p so that it extends to an elliptic operator on the double of X. Denote the extensions to $2X$ by the same letters as before. Then for $j = 1, 2$ use the Mellin transformation formula

$$\Gamma(\lambda) R_j^{-2\lambda} = \int_0^\infty t^{\lambda-1} e^{-tR_j^2} dt$$

to identify the value at $\lambda = 0$ of the analytic extension of the Schwartz kernel of $R_j^{-\lambda}$ on the diagonal with the coefficient of t^0 in the asymptotic expansion as $t \searrow 0$ of the Schwartz kernel of $e^{-tR_j^2}$ on the diagonal. Remember that $R_2 = Q_2$, and observe that Q_1^2 and R_1^2 are conjugate via v, so the pointwise trace of their heat kernels is the same. $\square$

Such a formula for the local term is not surprising in index theory. Our point is getting it without having to construct heat kernels for Φ-operators. In this respect the approach via complex powers, which are already objects in the calculus, presents a great advantage.

Although the local term (15) is smooth on X up to the boundary as a Φ-density, its integral might in principle diverge since as an usual density on X it has a singularity of order 3 at $x = 0$. Thus, we cannot set directly $z = 0$ in the above evaluations. To prove Theorem 2 we must identify the boundary term with the adiabatic limit of the eta invariant and show that the index density is integrable in a restricted sense. We will do this in Section 5.

However, we can prove directly that when A is a twisted Dirac operator corresponding to the metric g^X, and with twisting bundle T with metric and connection constant in x in a neighborhood of the boundary, the local term in the index formula equals the integral of the Atiyah-Singer density $\hat{A}(X, g^X) \operatorname{ch}(T)$ *without regularization.* Indeed,

Proposition 17. *The Riemannian curvature R of (X, g^X) induces a smooth 2-form on X with values in* $\mathrm{End}({}^{\Phi}TX)$ *down to $x = 0$. Thus $\hat{A}(X, g^X)$ and* $\mathrm{ch}(T)$ *are smooth forms on X.*

Proof. Let $e_1, \ldots, e_m$ be a local orthonormal frame in the fibers of φ. It is straightforward to compute $R(g^X)$ using the local orthonormal Φ-vector fields $x^2\partial_x$, $x\partial_\theta$ (lifted to ∂X using the connection involved in the definition of g^X), $e_1, \ldots, e_m$. We observe that while for instance $R(\partial_x, e_i)\partial_\theta$ diverges like $1/x$ as $x \to 0$, the induced action of R on ${}^{\Phi}TX$ is smooth down to $x = 0$. To conclude that $\hat{A}(X, g^X)$ is smooth it is enough to prove that $\mathrm{tr}(R^k)$ is smooth down to $x = 0$ for all $k \in \mathbb{N}$. Of course it does not matter for the trace if we view the 2-form R as acting on TX or on ${}^{\Phi}TX$, so the conclusion follows. $\mathrm{ch}(T)$ is obviously smooth in x (it is in fact constant in x in a neighborhood of $x = 0$). $\square$

5. The index of first-order Φ-differential operators

For the rest of the paper we assume that $Y = S^1$ and that A satisfies the hypothesis of Theorem 2.

Lemma 18. *The operator $x^2\partial_x - x/2$, defined on $\mathcal{C}_c^\infty(X^\circ, \mathcal{E})$ in a neighborhood of ∂X, is skew-symmetric with respect to the metric g^X.*

Proof. From (1), the volume form dg^X equals $\frac{dx}{x^3}d\theta dg^F$. Let $\phi_1, \phi_2 \in \mathcal{C}_c^\infty((0,\varepsilon) \times \partial X, \mathcal{E})$. Then

$$\begin{aligned}
&\int_X \big\langle (x^2\partial_x - x/2)\phi_1, \phi_2 \big\rangle + \big\langle \phi_1, (x^2\partial_x - x/2)\phi_2 \big\rangle \, dg^X \\
&= \int_{\partial X \times (0,\varepsilon)} (\partial_x - 1/x)\langle \phi_1, \phi_2 \rangle \frac{dx}{x} d\theta dg^F \\
&= \int_{\partial X} d\theta dg^F \int_{(0,\varepsilon)} \partial_x \left(\langle \phi_1, \phi_2 \rangle / x \right) dx \\
&= 0
\end{aligned}$$

since $\langle \phi_1, \phi_2 \rangle$ is assumed to have compact support. $\square$

We compute directly from Proposition 8 the values of the map q on the operators involved in Theorem 2. Set

$$\Delta := \begin{bmatrix} D^*D & 0 \\ 0 & DD^* \end{bmatrix}$$

seen as a family of elliptic operators over the fibers of φ, and also as a constant formal series in x of such operators.

$$\begin{aligned}
q(\delta_x^2) &= \begin{bmatrix} \tau^2 + D^*D & -ix\tilde{\nabla}_{\partial_\theta}(D^*) \\ ix\tilde{\nabla}_{\partial_\theta}(D) & \tau^2 + DD^* \end{bmatrix} \\
q(A) &= \sigma \begin{bmatrix} i\xi - \frac{x}{2} + \tau & D^* \\ D & i\xi - \frac{x}{2} - \tau \end{bmatrix} \\
\mathcal{N}(A^*) &= \begin{bmatrix} -i\xi + \tau & D^* \\ D & -i\xi - \tau \end{bmatrix} \sigma^* \\
q(A^*) &= \begin{bmatrix} -i\xi + \frac{x}{2} + \tau & D^* \\ D & -i\xi + \frac{x}{2} - \tau \end{bmatrix} \sigma^* \qquad (17) \\
\mathcal{N}(AA^*) &= \sigma \left(\begin{bmatrix} \xi^2 + \tau^2 + D^*D & 0 \\ 0 & \xi^2 + \tau^2 + DD^* \end{bmatrix} \right) \sigma^* \\
q(AA^*) &= \sigma \left(\begin{bmatrix} \xi^2 + \tau^2 + D^*D & 0 \\ 0 & \xi^2 + \tau^2 + DD^* \end{bmatrix} \right. \\
&\quad \left. + x \begin{bmatrix} -i\xi + \tau & -i\tilde{\nabla}_{\partial_\theta}(D^*) \\ i\tilde{\nabla}_{\partial_\theta}(D) & -i\xi - \tau \end{bmatrix} + \frac{x^2}{4} \right) \sigma^*.
\end{aligned}$$

Lemma 19. *The operator A defined by* (2) *is a fully elliptic Φ-operator if and only if the family D is invertible.*

Proof. It is clear from (17) that $\mathcal{N}(AA^*) = \sigma(\xi^2 + \tau^2 + \Delta)\sigma^*$ is elliptic and non-negative as a 2-suspended operator; moreover it is invertible for each value of the parameters $\tau, \xi \in \mathbb{R}$ (thus invertible as a suspended operator, see [14]) if and only if D is invertible. □

Lemma 20. [17, 18] *The differential operator $\delta_x \in \mathrm{Diff}^1(\partial X, \mathcal{E})$ is invertible for $0 < x < \varepsilon$ for some $\varepsilon > 0$.*

Proof. (sketch) We can view δ_x as an adiabatic family of operators (i.e., an adiabatic differential operator in the sense of [18]). The adiabatic normal operator of this family is invertible as in Lemma 19, so exactly like in Lemma 14 there exists an inverse $\mu_x \in \Psi_a^{-1}(\partial X, \mathcal{E})$ modulo $x^\infty \Psi_a^{-\infty}(\partial X, \mathcal{E})$:

$$\delta_x \mu_x = I + r_x$$

Now the residual adiabatic ideal $x^\infty \Psi_a^{-\infty}(\partial X, \mathcal{E})$ equals the space of rapidly vanishing families of smoothing operators on ∂X as $x \to 0$. Thus $r_x \to 0$ inside bounded operators as $x \to 0$. The conclusion follows for ε chosen small enough so that $\|r_x\| < 1, \forall x < \varepsilon$. □

Proposition 21. *The boundary term* $\widehat{\mathrm{Tr}}_\partial([A, \log x]B)$ *from the index formula* (14) *equals* $\frac{i}{2\pi}\log(\mathrm{hol}(\det D))$.

Proof. From (2) we know $q([A, \log x]) = x\sigma$. We claim that we can assume $\sigma = 1$. Let $S, T \in \mathcal{C}^\infty(X, \mathrm{End}(\mathcal{E})) \subset \Psi^0_\Phi(M, \mathcal{E})$ be such that $q(S) = \sigma, q(T) = \sigma^{-1}$. Since $q(ST) = 1$ it follows

$$\begin{aligned}\mathrm{Tr}(x^z[A, \log x]BQ_1^{-\lambda}) &= \mathrm{Tr}(x^z ST[A, \log x]BQ_1^{-\lambda}) + O(z^0)\\ &= \mathrm{Tr}(x^z T[A, \log x]BQ_1^{-\lambda}S) + O(z^0).\end{aligned}$$

Observe that $q(T[A, \log x])$ and $q(BQ_1^{-\lambda}S)$ do not contain σ anymore; on the other hand the term regular in z at $z = 0$ does not affect the residue, which proves our claim. From now on, we assume that $\sigma = 1$ in (2), so $q([A, \log x]) = x$. From Lemma 13,

$$\begin{aligned}&\widehat{\mathrm{Tr}}_\partial([A, \log x]B)\\ &= \frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2} \mathrm{Tr}\left(q([A, \log x]BQ_1^{-\lambda})_{[-2]}\right) d\tau d\xi d\theta_{|\lambda=0} \qquad (18)\\ &= \frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2} \mathrm{Tr}\left(q(A^*) * q(AA^*)^{-\frac{\lambda}{2}-1})\right)_{[-1]} d\tau d\xi d\theta_{|\lambda=0}.\end{aligned}$$

The plan is to use Eq. (17) for $q(A^*), q(AA^*)$ with $\sigma = 1$, as well as the explicit form of the leading terms in (11). There are three types of terms occurring in (18) as explained below and we write accordingly

$$\widehat{\mathrm{Tr}}_\partial([A, \log x]B) = (I(\lambda) + II(\lambda) + III(\lambda))_{|\lambda=0}.$$

The terms of type I. First there are those terms (called of type I) where $q(A^*)$ and $\mathcal{N}((AA^*)^{-\lambda/2-1})$ are composed according to the product rule (11). Since $\mathcal{N}((AA^*)^{-\lambda/2-1})$ is constant in x, the term coming from $ix\partial_\xi \otimes x\partial_x$ vanishes. Since $\frac{\partial\mathcal{N}(A^*)}{\partial\tau}$ is constant, the integrands corresponding to $x\partial_\tau \otimes \tilde{\nabla}_{\partial_\theta}$ is an exact form whenever it is of trace class by the following lemma, so the term coming from it also vanishes by unique continuation.

Lemma 22. *Let* $\pi : M \to Y$ *be a locally trivial fiber bundle with compact fiber* F *with a fixed connection,* ∂_θ *a vector field on the base lifted to* M *and* ∇ *a covariant derivative in a vector bundle* $\mathcal{E} \to M$. *Let* P *be a family of pseudodifferential operators of order* k *on the fibers of* π *with coefficients in* $\mathcal{E}$, *with* $k < -\dim F$. *Then*

$$\mathrm{Tr}(\nabla_{\partial_\theta}(P)) = \partial_\theta \,\mathrm{Tr}(P).$$

Proof. First notice that the family P is of trace class. Choose a Riemannian metric on the fibers and a hermitian metric on $\mathcal{E}$. The statement is linear in P; by decomposing $P = P_1 + iP_2$ with P_1, P_2 self-adjoints we can assume that P is self-adjoint. Let $\{s_j(b)\}_{j\in\mathbb{N}}$ be a smooth family of orthonormal eigensections of P_p for $b \in M$ of eigenvalue $\lambda_j(b)$. Evidently

$$(\nabla_{\partial_\theta}(P))s_j(b) = (\partial_\theta \lambda_j(b))s_j(b)$$

which implies the lemma. □

The non-vanishing terms of type I come from $1 \otimes 1$, i.e., involving $\frac{x}{2}\mathcal{N}(AA^*)^{-\lambda/2-1}$ (we call this term of type I_1) and from $-ix\partial_\xi \otimes (\tau\partial_\tau + 2\xi\partial_\xi)$, namely involving

$$\frac{x}{i}\frac{\partial\mathcal{N}(A^*)}{\partial\xi}\left(\tau\frac{\partial\mathcal{N}(AA^*)^{-\lambda/2-1}}{\partial\tau} + 2\xi\frac{\partial\mathcal{N}(AA^*)^{-\lambda/2-1}}{\partial\xi}\right)$$

(this term does not vanish as above, because of the τ and ξ factors; we call it of type I_2). For the term I_1, using (17) and polar coordinates in the (τ, ξ) plane we get

$$\begin{aligned} I_1(\lambda) &= \frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2}\frac{1}{2}\operatorname{Tr}\left(\mathcal{N}(AA^*)^{-\lambda/2-1}\right)d\theta d\tau d\xi \\ &= \frac{1}{4\pi\lambda}\int_{S^1}\operatorname{Tr}\left(\Delta^{-\frac{\lambda}{2}}\right)d\theta. \end{aligned} \tag{19}$$

Thus at $\lambda = 0$ we get the average of the logarithm of the determinant of the family $\Delta^{1/2}$ indexed by $e^{i\theta} \in S^1$. Although in the end this term will cancel away, it is worth recalling the definition of the determinant, not to confuse with the determinant line bundle with connection $(\det D, d+\omega^{BF})$ defined in Section 2. The zeta function of $\Delta^{1/2}$ is by definition

$$\zeta(\lambda) := \operatorname{Tr}(\Delta^{-\lambda/2}).$$

This function is analytic for $\Re(\lambda) > m$ (recall that m is the dimension of the fibers of φ), extends meromorphically to $\mathbb{C}$ and is regular at $\lambda = 0$; the logarithm of the determinant of $\Delta^{1/2}$ is defined as $-\zeta'(0)$. Coming back to I_1, this derivative clearly equals the finite part at $\lambda = 0$ of $-\frac{1}{\lambda}\operatorname{Tr}(\Delta^{-\frac{\lambda}{2}})$.

Similarly we get

$$
\begin{aligned}
I_2(\lambda) &= \frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2} \mathrm{Tr}\left(\frac{1}{i}\frac{\partial q(A^*)}{\partial \xi}\right.\\
&\qquad \left.\left(\tau\frac{\partial}{\partial \tau}+2\xi\frac{\partial}{\partial \xi}\right)\mathcal{N}(AA^*)^{-\frac{\lambda}{2}-1}\right)d\theta d\tau d\xi \\
&= \frac{3}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2}\mathrm{Tr}(\tau^2+\xi^2+\Delta)^{-\frac{\lambda}{2}-1}d\theta d\tau d\xi \\
&= \frac{3}{2\pi\lambda}\int_{S^1}\mathrm{Tr}\left(\Delta^{-\frac{\lambda}{2}}\right)d\theta
\end{aligned}
\tag{20}
$$

We used integration by parts in τ and ξ, Eq. (17) with $\sigma = 1$ and then polar coordinates in the (τ, ξ) plane.

The terms of type II**.** The second type of terms in (18) come from $1\otimes 1$ applied to $\mathcal{N}(A^*)$ and to the coefficient of x in $(\xi^2+\tau^2+\Delta)^{-\lambda/2-1}$, where the power is taken with respect to the product (11). The diagonal matrix $(\xi^2+\tau^2+\Delta)_{[-1]}^{-\lambda/2-1}$ is not explicitly computable; however we can compute $\mathrm{Tr}\left(\mathcal{N}(A^*)(\xi^2+\tau^2+\Delta)_{[-1]}^{-\lambda/2-1}\right)$ as if all the operators involved commute because of two facts:

- The diagonal of $\mathcal{N}(A^*)$ is made of central elements modulo x.
- The partial derivatives of $\xi^2+\tau^2+\Delta$ with respect to ξ and τ, are central elements in $\Psi_\Phi^{\mathbb{Z}}(X)/x^\infty\Psi_\Phi^{\mathbb{Z}}(X)$ modulo x.

$$
\begin{aligned}
II(\lambda) &= \frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2}\mathrm{Tr}\left(\mathcal{N}(A^*)(\xi^2+\tau^2+\Delta)_{[-1]}^{-\lambda/2-1}\right)d\tau d\theta d\xi \\
&= \frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2}\mathrm{Tr}\left(\left(-i\xi+\tau\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right)\frac{\left(-\frac{\lambda}{2}-1\right)\left(-\frac{\lambda}{2}-2\right)}{2}\right.\\
&\qquad\left.\left(\frac{4}{i}\xi(\tau^2+2\xi^2)+\frac{2}{i}\tau\tilde{\nabla}_{\partial_\theta}(\Delta)\right)(\xi^2+\tau^2+\Delta)^{-\frac{\lambda}{2}-3}\right)d\tau d\xi d\theta
\end{aligned}
$$

(the coefficient $\left(\frac{\lambda}{2}+1\right)\left(\frac{\lambda}{2}+2\right)$ is obtained by formally assuming that $-\frac{\lambda}{2}-1$ is a non-negative integer). We first eliminate the terms which are odd in ξ or τ and thus vanish after integration. The term containing $\tau^2\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]\tilde{\nabla}_{\partial_\theta}(\Delta)$ is also seen to vanish because the traces on E^+ and E^-

cancel each other. We are left with the term containing $-4\xi^2\tau^2 - 8\xi^4$.

$$
\begin{aligned}
II(\lambda) &= -\frac{1}{(2\pi)^2}\frac{\left(-\frac{\lambda}{2}-1\right)\left(-\frac{\lambda}{2}-2\right)}{2}\\
&\qquad\int_{S^1\times\mathbb{R}^2} 4(\xi^2\tau^2+2\xi^4)\operatorname{Tr}(\xi^2+\tau^2+\Delta)^{-\frac{\lambda}{2}-3}d\tau d\xi d\theta\\
&= -\frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2}(1+6)\operatorname{Tr}(\xi^2+\tau^2+\Delta)^{-\frac{\lambda}{2}-1}\\
&= -\frac{7}{4\pi\lambda}\int_{S^1}\operatorname{Tr}(\Delta)^{-\frac{\lambda}{2}}d\theta.
\end{aligned}
\tag{21}
$$

(we integrated by parts in τ and ξ and then used polar coordinates in the plane (τ,ξ).)

The terms of type III**.** These are the terms coming from $1\otimes 1$ applied to $\mathcal{N}(A^*)$ and to the coefficient of x in $q(AA^*)^{-\lambda/2-1}$ where now the power is with respect to the product of formal power series in x of fiberwise operators:

$$
\begin{aligned}
III(\lambda) = \frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2} &\operatorname{Tr}\big(\mathcal{N}(A^*)\\
&\left(\mathcal{N}(AA^*)+x\begin{bmatrix} -i\xi+\tau & -i\tilde{\nabla}_{\partial_\theta}(D^*)\\ i\tilde{\nabla}_{\partial_\theta}(D) & -i\xi-\tau\end{bmatrix}\right)^{-\frac{\lambda}{2}-1}\big)_{[-1]}d\tau d\xi d\theta
\end{aligned}
$$

Again, it is impossible to compute the integrand before taking the trace; however, the suspended operators $\mathcal{N}(A)$ and $\mathcal{N}(A^*)$ commute (Nota bene, with respect to the product of suspended algebra, not the product (11)). Indeed, under the assumption $\sigma=1$, $\mathcal{N}(A)$ and $\mathcal{N}(A^*)$ consist of the same symmetric operator plus a scalar multiple of the identity. Thus the leading factors $\mathcal{N}(A^*)$ and $\mathcal{N}(AA^*)$ also commute so the trace behaves as if all operators involved commuted. We get the following contribution to (18):

$$
\begin{aligned}
III(\lambda) = \frac{\left(-\frac{\lambda}{2}-1\right)}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2}&\operatorname{Tr}\left(\begin{bmatrix}-i\xi+\tau & D^*\\ D & -i\xi-\tau\end{bmatrix}\right.\\
&\left.(\xi^2+\tau^2+\Delta)^{-\frac{\lambda}{2}-2}\begin{bmatrix}-i\xi+\tau & -i\tilde{\nabla}_{\partial_\theta}(D^*)\\ i\tilde{\nabla}_{\partial_\theta}(D) & -i\xi-\tau\end{bmatrix}\right)d\tau d\xi d\theta.
\end{aligned}
$$

The middle term is a diagonal matrix. Let us look first at the terms coming from the diagonal entries in the first and third matrix, disregarding those which are odd in τ or ξ. They give

$$
\begin{aligned}
III_1(\lambda) &= \frac{\left(-\frac{\lambda}{2}-1\right)}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2}(-\xi^2+\tau^2)\operatorname{Tr}(\xi^2+\tau^2+\Delta)^{-\frac{\lambda}{2}-2}d\tau d\xi d\theta\\
&= 0
\end{aligned}
$$

by symmetry in τ and ξ. Finally let us compute the contribution coming from anti-diagonal entries. Again we use polar coordinates in the (τ,ξ) plane.

$$
\begin{aligned}
III_2(\lambda) &= \frac{i}{(2\pi)^2}\left(-\frac{\lambda}{2}-1\right)\int_{S^1\times\mathbb{R}^2}\\
&\quad\Big[\operatorname{Tr}\left(D^*(\xi^2+\tau^2+DD^*)^{-\frac{\lambda}{2}-2}\tilde{\nabla}_{\partial_\theta}(D)\right)\\
&\quad-\operatorname{Tr}\left(D(\xi^2+\tau^2+D^*D)^{-\frac{\lambda}{2}-2}\tilde{\nabla}_{\partial_\theta}(D^*)\right)\Big]\,d\tau d\xi d\theta\\
&= -\frac{i}{4\pi}\int_{S^1}\Big(\operatorname{Tr}\left(D^*(DD^*)^{-\frac{\lambda}{2}-1}\tilde{\nabla}_{\partial_\theta}(D)\right)\\
&\quad-\operatorname{Tr}\left(D(D^*D)^{-\frac{\lambda}{2}-1}\tilde{\nabla}_{\partial_\theta}(D^*)\right)\Big)\,d\theta\\
&= -\frac{i}{2\pi}\int_{S^1}\omega^{BF}(\lambda)d\theta.
\end{aligned}
\tag{22}
$$

In the last equality we used the identity

$$
\begin{aligned}
0 &= -\frac{2}{\lambda}\int_{S^1}\partial_\theta\operatorname{Tr}(DD^*)^{-\frac{\lambda}{2}}d\theta\\
&= -\frac{2}{\lambda}\int_{S^1}\operatorname{Tr}(\tilde{\nabla}_{\partial_\theta}(DD^*)^{-\frac{\lambda}{2}})d\theta && \text{by Lemma 22}\\
&= \int_{S^1}\Big(\operatorname{Tr}\left(D^*(DD^*)^{-\frac{\lambda}{2}-1}\tilde{\nabla}_{\partial_\theta}(D)\right)\\
&\quad+\operatorname{Tr}\left(D(D^*D)^{-\frac{\lambda}{2}-1}\tilde{\nabla}_{\partial_\theta}(D^*)\right)\Big)\,d\theta
\end{aligned}
$$

The terms (19), (20) and (21) cancel so Proposition 21 follows from (22) specialized at $\lambda=0$ and from (4). □

By Theorem 6 the quantity computed in Proposition 21 equals half the adiabatic limit (the limit as x tends to 0) of the eta invariant of the family δ_x. To complete the proof of the index Theorem 2 we show now the integrability of the index density.

Proposition 23. *Under the assumptions of Theorem* 2*, the local index density* $\mathrm{tr}(r_1(0) - r_2(0))$ *is a smooth multiple of* $1/x$*. Moreover,*

$$\lim_{\varepsilon\to 0} \int_{X\cap\{x\geq\varepsilon\}} \mathrm{tr}(r_1(0) - r_2(0))$$

exists and gives the $\overline{AS}$ *term in the index formula without regularization with* x^z*.*

Proof. By Corollary 12 we know that

$$\mathrm{tr}(r_j(0)) \sim_{x\to 0} \frac{1}{(2\pi)^2} \left(\int_{\mathbb{R}^2} \mathrm{tr}\, q(R_j^{-\lambda}) d\xi d\tau \right) \frac{d\theta dx}{x^3} \tag{23}$$

has a Laurent expansion at $x = 0$ with a possible singularity of order 3. Thus we first want to show that the coefficients of x^{-3}, x^{-2} in $\mathrm{tr}(r_1(0)) - \mathrm{tr}(r_2(0))$ vanish. Since tr is invariant under conjugation by linear isomorphisms, we can replace the operator U near $x = 0$ with

$$P := (x^2\partial_x - \frac{x}{2}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \delta_x.$$

We have

$$q(PP^*) = \xi^2 - ix\xi + \frac{x^2}{2} + q(\delta_x^2) + x\tau \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$q(P^*P) = \xi^2 - ix\xi + \frac{x^2}{2} + q(\delta_x^2) - x\tau \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From this we see that $\mathcal{N}(PP^*)^{-\lambda} = \mathcal{N}(P^*P)^{-\lambda}$, so the coefficient of x^{-3} vanishes. Also, $(q(PP^*)^{-\lambda} - q(P^*P)^{-\lambda})_{[-1]}$ is odd in τ hence vanishes after integration. Therefore (23) proves the first part of the proposition.

It follows that $x\rho(x)$ is smooth for $x \in [0, \varepsilon)$, where

$$\rho(x) := \int_{\partial X\times\{x\}} \mathrm{tr}(r_1(0) - r_2(0)).$$

Therefore

$$\int_0^\varepsilon \int_{\partial X\times\{x\}} x^z\, \mathrm{tr}(r_1(0) - r_2(0))$$

is holomorphic in z for $\Re(z) > 0$. By Corollary 15 this integral is also regular at $z = 0$, so the coefficient of x^{-1} in the asymptotic expansion of $\rho(x)$ at $x = 0$ vanishes. Thus, ρ is smooth on $[0, \varepsilon)$ as claimed. □

By computing explicitly traces of star-products as above, we could show that the integral along the fibers of the index density is smooth on $S^1 \times [0, \varepsilon)$. It seems reasonable to ask if the index density itself is smooth down to $x = 0$ (as in the case of Dirac operators), however we were unable to prove or to disprove this fact.

6. Link with d-geometry

In the case of Dirac operators our result (Corollary 3) is related to Vaillant's index formula [21]. Consider the so-called d-metric on X given by $g_d := x^2 g^X$ where g^X is the product Φ-metric (1). Then the Dirac operator D_d corresponding to g_d, acting in $L^2(X, \Sigma, g_d)$ with domain $\mathcal{C}_c^\infty(X, \Sigma)$, is isometric to $x^{-1/2}Ax^{-1/2}$ acting in $L^2(X, \Sigma, g^X)$ with the same domain, where A is the Dirac operator from Corollary 3. By Lemma 14 the L^2-nullspace of A is isomorphic to the L^2-nullspace of $x^{-1/2}Ax^{-1/2}$ via the map of multiplication by $x^{1/2}$. In particular, we deduce that $\operatorname{index}(D_p) = \operatorname{index}(A)$. The (extended) L^2 index of D_p was computed by Vaillant for arbitrary fibrations and for possibly non-invertible normal operators, under the assumption that the fiberwise kernels form a vector bundle. Thus Corollary 3 is a particular case of his result. Note that our method of proof, unlike the heat kernel technique of Vaillant, applies to operators which are not of Dirac type, see Theorem 2.

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