ODD PFAFFIAN FORMS

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Abstract. On any odd-dimensional oriented Riemannian manifold we define a volume form, which we call the odd Pfaffian, through a certain invariant polynomial with integral coefficients in the curvature tensor. We prove an intrinsic Chern-Gauss-Bonnet formula for incomplete edge singularities in terms of the odd Pfaffian on the fibers of the boundary fibration. The formula holds for product-type model edge metrics where the degeneration is of conical type in each fiber, but also for general classes of perturbations of the model metrics. The same method produces a Chern-Gauss-Bonnet formula for complete, non-compact manifolds with fibered boundaries in the sense of Mazzeo-Melrose and perturbations thereof, involving the odd Pfaffian of the base of the fibration. We deduce the rationality of the usual Pffaffian form on Riemannian orbifolds, and exhibit obstructions for certain metrics on a fibration to be realized as the model at infinity of a flat metric with conical, edge or fibered boundary singularities.

1. Introduction

Gauss-Bonnet formulas in singular geometric contexts abound in mathematical literature, we mention here for instance [1, 3, 7, 11, 12, 13, 14, 16, 22, 24, 25, 28]. With a few notable exceptions, most of those theorems treat the case of singular sets embedded in a smooth Riemannian manifold M, typically $M = \mathbb{R}^n$, since by the Nash embedding theorem all Riemannian manifolds are isometrically embeddable in some euclidean space. In this article we look at a different type of degeneration, for which the techniques of the "embedded" situation do not apply. Namely, we consider a compact differentiable manifold M with boundary, endowed with a Riemannian metric which is smooth in the interior and degenerates at the boundary following certain precise patterns. Examples of such degenerate metrics include the so called *incomplete edge metrics*, for instance any Riemannian metric in the complement of a submanifold, and also the *fibered boundary metrics*, a class of complete metrics including the generalized Taub-NUT metrics on $\mathbb{R}^4$.

Double forms and the odd Pfaffian. We set the stage with our own algebraic treatment of the Gauss-Bonnet formula on compact oriented manifolds (M^{2k}, g) using the formalism of double forms:

$$(2\pi)^k\chi(M) = \int_M \mathrm{Pf}(g), \qquad\qquad \mathrm{Pf}(g) = \frac{1}{k!}\mathcal{B}_g\left((R^g)^k\right).$$

Here $R^g \in \Lambda^2(M)\otimes\Lambda^2(M)$ is the curvature form of g, a double form of bi-degree $(2,2)$, and $\mathcal{B}_g$ is the Berezin integral, or contraction with the volume form of g in the second component. When M has a nonempty boundary (N,h), essentially as a consequence of the second Bianchi identity we isolate a correction term when the metric is not of product-type near the boundary:

$$(2\pi)^k\chi(M) = \int_M \mathrm{Pf}(g) - \sum_{j=0}^{k-1}\frac{(-1)^{k+j}(2k-2j-3)!!}{j!(2k-2j-1)!}\int_{\partial M}\mathcal{B}_h\left((R^h)^j\wedge \mathrm{II}^{2k-2j-1}\right). \tag{1.1}$$

Date: July 3, 2018.
2010 *Mathematics Subject Classification.* Primary 58A10, 53C05; Secondary 57R18.
Partially supported by the CNPq Universal Project.
Partially supported by the CNCS project PN-III-P4-ID-PCE-2016-0330.

In this formula $\mathrm{II} \in \Lambda^1(N) \otimes \Lambda^1(N)$ is the second fundamental form of the boundary, a double form of bi-degree $(1,1)$, $\mathcal{B}_h$ is the Berezin integral with respect to h, and

$$(-1)!! := 1, \qquad\qquad (2n-1)!! := 1 \cdot 3 \cdot \ldots \cdot (2n-1) \text{ for } n \geq 1.$$

Of course, in coordinates this coincides with the correction term of the original formulae of Allendoerfer-Weil [3] and Chern [8, 9]. This compact algebraic way of writing the Gauss-Bonnet integrand on the boundary is well-suited for generalizations.

Motivated by (1.1), we define the *odd Pfaffian form* of a $2k-1$-dimensional Riemannian manifold (N,h) in terms of the curvature form $R^h \in \Lambda^2 \otimes \Lambda^2$.

Definition 1.1. For every oriented $2k-1$-dimensional Riemannian manifold (N,h) define

$$\mathrm{Pf}^{\mathrm{odd}}(h) := \sum_{j=0}^{k-1} (-1)^{k+j}(2k-2j-3)!!\mathcal{B}_h\left(\frac{(R^h)^j \wedge h^{2k-1-2j}}{j!(2k-2j-1)!}\right) \in \Lambda^{2k-1}(N).$$

In any orthonormal frame, $\mathrm{Pf}^{\mathrm{odd}}$ is a polynomial with integral coefficients in the entries of the curvature form R. Up to a constant, this form appears already, in a different presentation, in the work of Albin [1, Eq. (7.12)] as the boundary correction term in the Gauss-Bonnet formula for scattering metrics. It consists of a linear combination with integral coefficients of the Lipschitz-Killing curvatures (Definition 4.3). As explained in Section 4, the odd Pfaffian is in fact the transgression of the Pfaffian for any slice $\{r\} \times N$ on the cone $(-\epsilon, 0) \times N$ with the metric $dr^2 \oplus r^2 h$.

Edge singularities. The first type of metric analyzed here are the incomplete edge metrics. This means we have an (oriented) compact manifold with boundary M together with a fibration structure of the boundary $\pi : \partial M \to B$ over a compact manifold B. Fix a boundary-defining function r for the boundary. The (singular) metric in a collar neighborhood of $\partial M = \{r = 0\}$ has the form

$$g = dr^2 \oplus g(r), \qquad\qquad g(r) = r^2 g^V \oplus \pi^* g^B \tag{1.2}$$

where g^B is a metric on B, g^V is a Riemannian metric on the fibers and the splitting is induced by an Ehresmann connection. Even in this first analysis we allow g^V to vary with r but still converging to some true metric at $r = 0$.

We prove that a Gauss-Bonnet formula holds on such manifolds and we compute the contribution of the singular locus ∂M in terms of the geometric data, essentially the Pfaffian of the base and the odd Pfaffian of the fibers. Due to its importance in geometric applications, we review the (perturbed) conical case separately (see Theorem 4.6).

Theorem 1.2. *Let (M^{2k}, g) be a manifold with edge singularities.*

(a) *If* $\dim(B)$ *is odd,*

$$\chi(M) = \frac{1}{(2\pi)^k}\int_M \mathrm{Pf}^g.$$

(b) *If* $\dim(B)$ *is even,*

$$(2\pi)^k\chi(M) = \int_M \mathrm{Pf}^g - \int_B\left(\mathrm{Pf}(g^B)\int_{\partial M/B}\mathrm{Pf}^{\mathrm{odd}}(g^V)\right).$$

When we allow horizontal variations of the metric, i.e. g^B varies with r, we obtain certain additional terms (see Theorem 5.11).

The computation is based on two observations. First, the second fundamental form of a slice is the Lie derivative of the metric in the direction of the normal geodesic flow ∂_r. Secondly, we describe explicitly the decomposition of the curvature form of a Riemannian submersion into its horizontal, mixed and vertical components with respect to the second variable when seen as a double form.

Manifolds with fibered boundaries. The same method used for edge metrics leads to a Gauss-Bonnet formula for a different type of degeneracy. Following Mazzeo and Melrose [18], a non-compact Riemannian manifold (M, g) is called with *fibered boundary* if it has a finite number of ends which are modeled on $(1, \infty) \times N$ with the metric

$$g := dr^2 \oplus g^V \oplus r^2 \pi^* g^B$$

for $r \gg 1$. We assume here that $N \to B$ is a fiber bundle with a fixed Ehresmann connection with respect to which g is given. It is not hard to see that such a metric is complete. (These metrics were studied in depth by Vaillant in [26] under the name ϕ-metrics.)

Theorem 1.3. *Let (M^{2k}, g) be a manifold with fibered boundary. Denote by F a generic fiber and by f its dimension.*

(a) *If b is even,*

$$\chi(M) = \frac{1}{(2\pi)^k} \int_M \mathrm{Pf}^g.$$

(b) *If b is odd,*

$$(2\pi)^k \chi(M) = \int_M \mathrm{Pf}^g + (2\pi)^{f/2} \chi(F) \int_B \mathrm{Pf}^{\mathrm{odd}}(g^B). \tag{1.3}$$

Compared with Theorem 1.2 there are two formal differences: the odd Pfaffian appears now in the base, not in the fibers; and the sign in front of the transgression has changed.

The Gauss-Bonnet problem for fibered boundary metrics was previously studied by Albin [1] and also by Dai-Wei [13]. Theorem 1.3 can be seen as an extension of their partial results. Albin gives a formula in the case where either the fiber or the base of the boundary fibration reduce to a point, while for $]dim(M) = 4$, Dai and Wei give the formula when the fiber is a point, i.e., for "large conical" metrics, better known as scattering metrics by the Melrose school. Note that Dai-Wei also state a formula in the general case, claiming the vanishing of the transgression term from (1.3). This claim holds true for even-dimensional B, but is incorrect when the base is odd-dimensional, as noted also in [30]. (They apply this result in dimension four when the fiber is a circle, hence their results concerning Hitchin-Thorpe inequalities on blow-ups of the Taub-NUT space are not affected by this issue.)

Orbifolds. In the second part of the article we look at perturbations of the model metrics g described in (1.2). We show that if the perturbations of g are of second order, in a sense made precise in Def. 7.6, the formulæ from Theorems 1.2 and 1.3 remain valid.

A natural example of second-order perturbation of a model edge metric is the complement of a submanifold B in a Riemannian manifold M when one lifts the original metric to the oriented blow-up of B. The formula in this case reflects a basic topological fact:

$$\chi(M \setminus B) = \chi(M) - \chi(B).$$

The situation becomes more interesting when we blend in isometric actions of finite groups. If M is a Riemannian orbifold with singularities locally modeled on quotients of type N/G where G acts freely on $N \setminus \mathrm{Fix}_G(N)$ and $\mathrm{Fix}_G(N)$ is a smooth submanifold locus, we obtain the following Gauss-Bonnet formula for orbifolds:

Theorem 1.4. *Let $\hat{M}$ be a compact Riemannian orbifold with simple singularities of dimension $2k$ and let g be the Riemannian metric on $\hat{M} \setminus Z$. Then*

$$\chi(\hat{M}) = \frac{1}{(2\pi)^k} \int_{\mathrm{Int}\, \hat{M}} \mathrm{Pf}^g - \sum_{Z_i \in \mathrm{Fix}(\hat{M})} \frac{\chi(Z_i)}{|G_i|} \tag{1.4}$$

where $\mathrm{Fix}(\hat{M})$ is the set of connected components of the singular locus of $\hat{M}$.

Perturbations of the model degenerate metrics and transgressions. To our best knowledge, Chern-Gauss-Bonnet formulæ for incomplete edge metrics and for fibered boundary metrics are new. In fact, in the context set forth in this paper, besides the thesis paper of Albin which we already commented upon, previous results were also obtained by Rosenberg [25] whose main statement can be seen as particular case of Theorem 1.3 (see also the "conical" Gauss-Bonnet Theorem 4.6).

In our view, the main contribution of this note is being able to extend the results from model metrics to general classes of perturbations.

It turns out that when one deals with the (product-type) model metrics, one can take advantage of certain "symmetries" in order to perform the computations, like being able to keep track of the different components of the curvature form and second fundamental form. This does not seem to be case when allowing perturbations and a direct, computational approach raises multiple difficulties.

In compensation, properties of transgression forms are fundamental for the proofs given here and allow us to use "topological" arguments in places where the computation of geometric quantities seems overly complicated. We devote a first section to proving such properties, since they are not part of mainstream presentation of Chern-Weil theory.

Recall that on a given Euclidean vector bundle $E \to B$ of rank $2k$ endowed with two metric connections ∇_1, ∇_2 there exists a canonical form $\mathrm{TPf}(\nabla_1, \nabla_2)$ that satisfies:

$$\mathrm{Pf}(\nabla_1) - \mathrm{Pf}(\nabla_2) = d\mathrm{TPf}(\nabla_1, \nabla_2).$$

It is known since Chern [9] that the boundary integrand in the standard Gauss-Bonnet Theorem can be described as such a transgression form. So at first it might seem unremarkable that the correction term in Gauss-Bonnet Theorem for first order perturbations (see below) of the model metric is a transgression form integrated over the boundary. However, one should keep in mind that due to the degeneracy of the metric, there is *a priori* no well-defined connection along the singular locus, let alone two of them.

We analyze perturbations of the model degenerate metrics, both for incomplete edge metrics and for complete fibered boundary metrics. The methods to treat the two cases are similar and we only outline here the treatment of the non-complete case. One natural approach would be to follow the ideas first introduced by Melrose in the general context of the b-calculus [19, 20], and employ as background the edge tangent bundle, transferring all geometric structures onto it. However, since the edge tangent bundle is isomorphic (albeit non-canonically) to the tangent bundle, rather than relying explicitly on this natural notion we prefer to work here with an endomorphism $\varphi \in \mathrm{End}(TM)$ which has, given the choice of a boundary defining function r, the following expression in a collar neighborhood of ∂M:

$$\varphi(v, w) = (rv, w),$$

i.e., φ acts as multiplication by r on the vertical component of the fiber bundle $\partial M \to B$ and leaves the horizontal and the normal components unchanged. (Of course, the edge tangent bundle remains hidden behind the curtain.)

The endomorphism φ is an isomorphism in the interior but not at $r = 0$. It is easy to see that the model degenerate metric g has the property that the pull-back metric

$$g^{\varphi}(\cdot, \cdot) := g(\varphi^{-1}(\cdot), \varphi^{-1}(\cdot))$$

extends to a smooth metric on TM. Consequently, we consider perturbations $\tilde{g}$ of g that preserve this property. In fact, a perturbation $\tilde{g}$ of g is a degenerate metric that satisfies

$$\tilde{g}^{\varphi} = g^{\varphi} + \alpha(\cdot, \cdot)$$

for certain smooth symmetric bilinear form α which vanishes at least to order 1 at $r = 0$. We call the perturbation to be of order $j \geq 1$ if $\alpha \in O(r^j)$.

The main result that allows the investigation of Gauss-Bonnet formulas for perturbations of model metrics is the next theorem which should be compared with extension results for the Levi-Civita connection in the context of ϕ-geometry (see [26], Prop. 1.5).

Theorem 1.5. *Let* ∇^g, $\nabla^{\tilde{g}}$ *be the Levi-Civita connections of the degenerate metric* g *and a first order perturbation* $\tilde{g}$*. Then* $\varphi\nabla^g\varphi^{-1}$ *and* $\varphi\nabla^{\tilde{g}}\varphi^{-1}$ *extend to smooth connections on* TM*. If* $\tilde{g}$ *is a second-order perturbation, then the restriction of these connections to* $r = 0$ *coincide:*

$$\varphi\nabla^{\tilde{g}}\varphi^{-1}\big|_{r=0} = \varphi\nabla^g\varphi^{-1}\big|_{r=0}. \tag{1.5}$$

We use an "abstract" version of the Christoffel coefficients formula which reduces this theorem to proving the smooth extension at $r = 0$ of the Levi-Civita connection for the model metric g. It is exactly property (1.5) that allows one to conclude that Theorem 1.2 holds for second-order perturbations.

In Theorem 1.2, if we allow first-order perturbations of g, there appears an extra contribution at the boundary in the guise of a certain transgression form. For this kind of perturbations let:

$$\nabla^1 := \varphi\nabla^{\tilde{g}}\varphi^{-1}\big|_{r=0}, \qquad \nabla^0 := \varphi\nabla^g\varphi^{-1}\big|_{r=0}.$$

Then the following general Gauss-Bonnet formula holds:

Theorem 1.6. *Let* $\tilde{g}$ *be a first order perturbation of a model edge metric* $g = dr^2 \oplus r^2 g^V \oplus \pi^* g^B$*. Then*

$$(2\pi)^k \chi(M) = \int_M \mathrm{Pf}^{\tilde{g}} - \int_B \left(\mathrm{Pf}(g^B) \int_{\partial M/B} \mathrm{Pf}^{\mathrm{odd}}(g^V) \right) - \int_{\partial M} \mathrm{TPf}(\nabla^1, \nabla^0).$$

The form $\mathrm{Pf}(g^B)$ *is zero, by definition, when* $\dim B$ *is odd.*

The restriction ∇^0 has a particularly simple geometric description (see Corollary 7.2).

In the particular case when the degeneration is of first order with respect to a conical metric we are able to give a geometric expression for the boundary contribution in the spirit of the classical Gauss-Bonnet. Let

$$\mathcal{G}^{\partial M}_{j,2k-1} := \frac{1}{j!(2k-1-2j)!} \mathcal{B}_{g^\varphi}\left((R^N)^j \wedge (\mathrm{II}^g)^{2k-1-2j} \right).$$

where the second fundamental form II^g is defined via ∇^1 above.

Theorem 1.7. *For first order perturbations* g *of conical metrics* $dr^2 \oplus r^2 g^N$ *the following holds*

$$(2\pi)^k \chi(M) = \int_M \mathrm{Pf}^g - \sum_{j=0}^{k-1} (-1)^j (2j-1)!! \int_{\partial M} \mathcal{G}^{\partial M}_{k-1-j,2k-1}$$

Similar results, proved with the same techniques, hold for first and second order perturbations of manifolds with fibered boundary (see Section 8). One simple example of a second order perturbation of a flat metric is the catenoid.

Historical notes. The Gauss-Bonnet formula for polygonal surfaces embedded in Euclidean 3-space was found almost 200 years ago by Gauss, Binet and Bonnet. Our standard textbook formula for closed surfaces in R^3 linking the Euler characteristic with the integral of the Gaussian curvature was stated and proved by Walther von Dyck [27] at the end of the 19th century. The modern history of its generalizations can be found in the nice survey [29]. The integrand in higher dimensions was isolated in the 1920's by Heinz Hopf in the case of hypersurfaces in Euclidean space, while the validity of Hopf's formula for embedded manifolds of arbitrary codimension in R^n was independently proved in 1940 by Allendoerfer and Fenchel, building on work of Weyl. In 1943 Allendoerfer and Weil [3] not only proved the validity of Hopf's formula in the abstract (non-embedded) case, but also gave the correction term for a manifold with boundary. They went even further and produced a formula that is valid for a topological manifold with boundary which is a Riemannian polyhedron, i.e., boundary points have neighborhoods which are differentially modeled on convex cones in $\mathbb{R}^n$ and there exists a globally defined smooth Riemannian metric on the resulting differentiable polyhedron. Their theorem is in some sense at the crossroad of what we call embedded/non-embedded situation. Soon afterwards, S. S. Chern [8, 9] gave intrinsic proofs for compact smooth Riemannian manifolds, both with and without boundary. Chern's articles have been immensely influential. With regard to more modern developments, the generalization of the Allendoerfer-Weil theorem of R. Walter [28] on compact locally convex subsets of Riemannian manifolds anticipates the techniques coming from Geometric Measure Theory with applications to the integral geometry of subanalytic cycles promoted by J. Fu [16]. Ideas from stratified Morse theory have also been used successfully in the context of integral geometry of tamed sets [7]. More recently, an enhanced version of the Allendoerfer-Weil theorem was used by McMullen [22] to compute the volume of the moduli space of n-pointed Riemann surfaces of genus 0. Other important works related to the topic of this paper are cited in the bibliography.

Acknowledgements: The first named author had interesting discussions about the topic of the article with Jorge de Lira, Luciano Mari and Luquesio Jorge and for that he would like to thank them. He would like to particularly thank Vincent Grandjean who patiently listened to the crude ideas that finally took shape inhere.

2. The transgressions of the Pfaffian. General facts

We include here a series of general facts, more or less well-known, about the transgression of the Pfaffian. There exist various incarnations of the transgression form and one of the purposes of this section is to bring them under the same umbrella. These facts play the equally important role of simplifying the presentation in the sequel.

2.1. **Transgressions and connections.** Let $E \to M$ be an oriented Euclidean vector bundle of rank $2k$ over a manifold M. Every connection ∇ compatible with the metric gives rise to a closed form of degree $2k$ on M, the Pfaffian associated to the curvature tensor $F(\nabla) := d^{\nabla} \circ \nabla$, locally a skew-symmetric matrix of 2-forms. If $F(\nabla)_{ij} := \langle F(\nabla)s_j, s_i\rangle$ in a local orthonormal basis $\{s_1, \ldots, s_{2k}\}$ of E then

$$\mathrm{Pf}(\nabla) := \frac{1}{2^k k!} \sum_{\sigma \in S_{2k}} \epsilon(\sigma) F(\nabla)_{\sigma(1)\sigma(2)} \wedge \ldots \wedge F(\nabla)_{\sigma(2k-1)\sigma(2k)}. \tag{2.1}$$

In the next section we will define the Pfaffian intrinsically via double forms, proving its gauge independence. What is special about the Pfaffian compared to other invariant polynomials is that it vanishes in the presence of a non-zero parallel section in E.

Given a smooth path of metric connections $\alpha^{\nabla} := (\nabla^t)_{t\in[0,1]}$, one can construct a transgression form $\mathrm{TPf}(\alpha^{\nabla})$ which satisfies

$$d\mathrm{TPf}(\alpha^{\nabla}) = \mathrm{Pf}(\nabla^1) - \mathrm{Pf}(\nabla^0). \tag{2.2}$$

The construction goes as follows. On the oriented Euclidean vector bundle $\pi_2^*E \to [0,1] \times M$ (where $\pi_2 : [0,1] \times M \to M$ is the projection) consider the connection $\tilde{\nabla} := \frac{d}{dt} + \nabla^t$ which acts on a section $(s_t)_{t\in[0,1]}$ of π_2^*E as follows:

$$\tilde{\nabla} s_t = dt \otimes \frac{\partial s_t}{\partial t} + (\nabla^t s_t).$$

Consider the Pfaffian $\mathrm{Pf}(\tilde{\nabla})$ which is a closed form and use the homotopy formula for $H := \mathrm{id}_{[0,1]\times M}$ and $\mathrm{Pf}(\tilde{\nabla})$ to conclude that (2.2) is valid with

$$\mathrm{TPf}(\alpha^{\nabla}) := \int_{[0,1]} \mathrm{Pf}(\tilde{\nabla}),$$

the integral being over the fibers of the projection π_2.

Example 2.1. Suppose (M, g) is a Riemannian manifold with boundary of even dimension. Then the Euclidean vector bundle $TM\big|_{\partial M} \to \partial M$ is endowed with two metric connections. One is the Levi-Civita connection $\nabla^1 := \nabla^M$ on M and the other one is "the cylindrical connection" $\nabla^0 := d \oplus \nabla^{\partial M}$ where we use the splitting

$$TM\big|_{\partial M} = \mathbb{R}\nu \oplus T\partial M \tag{2.3}$$

induced by the unit normal ν. Notice that $\mathrm{Pf}(\nabla^0) = 0$ since ν is a parallel section and the curvature splits into a direct sum of factors, one of which is zero. We use the affine path of connections $\tilde{\nabla} := (1-s)\nabla^0 + s\nabla^1$ to construct the Chern transgression $\mathrm{TPf}(\nabla^M) = \mathrm{TPf}^g$ associated to the metric g. This is the form which appears in the Gauss-Bonnet formula.

If the splitting (2.3) is extended to a neighborhood U of ∂M (e.g. via minus the gradient of the distance function to ∂M) the identity $d\mathrm{TPf}^g = \mathrm{Pf}^g$ on U is valid on U.

Remark 2.2. For $-\alpha^\nabla$ defined via $-\alpha^\nabla(t) := \alpha^\nabla(1-t)$ one has:

$$\mathrm{TPf}(-\alpha^\nabla) = -\mathrm{TPf}(\alpha^\nabla).$$

Indeed, one uses the orientation-reversing diffeomorphism

$$[0,1] \times M \to [0,1] \times M, \qquad (t, m) \to (1-t, m)$$

while fiberwise integration is sensitive to the orientation.

Proposition 2.3. *For two smooth paths* α^∇ *and* β^∇ *of metric connections with* $\alpha^\nabla(i) = \beta^\nabla(i)$, $i = 0, 1$ *there exists a form* $\mathrm{TPf}(\alpha^\nabla, \beta^\nabla)$ *of degree* $2k-2$ *such that:*

$$\mathrm{TPf}(\alpha^\nabla) - \mathrm{TPf}(\beta^\nabla) = d\mathrm{TPf}(\alpha^\nabla, \beta^\nabla). \tag{2.4}$$

Proof. Let $\mathcal{A}$ be the space of affine connections compatible with the metric. It is an affine space modeled on $\Gamma(M; \Lambda^2 T^*M \otimes \mathrm{End}^-(E))$. Let $\square := [0,1] \times [0,1]$. Consider the smooth family of connections

$$\widetilde{\alpha\beta} : \square \to \mathcal{A}, \qquad \widetilde{\alpha\beta}(s,t) = (1-s)\alpha^\nabla(t) + s\beta^\nabla(t).$$

On the vector bundle $\pi_3^* E \to \square \times M$ (where $\pi_3 : \square \times M \to M$ is the projection) consider the connection $\hat{\nabla} := \partial_s + \partial_t + \alpha\beta(s,t)$ which acts on a smooth section $u : \square \to \Gamma(M; E)$ of $\pi_{1,2}^* E$ via

$$\hat{\nabla} s = du \otimes \frac{\partial u}{\partial s} + dt \otimes \frac{\partial u}{\partial t} + (1-s)\alpha^\nabla(t)(u(s,t)) + s\beta^\nabla(t)(u(s,t)).$$

Applying Stokes formula on $\square$ to the smooth closed form $\mathrm{Pf}(\hat{\nabla}) \in \Lambda^*(\square \times M)$ we obtain

$$-d\int_\square \mathrm{Pf}(\hat{\nabla}) = \int_{\partial\square} \mathrm{Pf}(\hat{\nabla})$$

where integration is really integration over the fibers of the projections $\square \times M \to M$ and $(\partial\square) \times M \to M$. Now $\partial\square$ consists of two constant paths of connections for $t = 0$ and $t = 1$, while for $s = 0$ and $s = 1$ by definition the integral on the right hand side gives the transgressions induces by α^∇ and β^∇. Taking into account the orientations, we get (2.4) with

$$\mathrm{TPf}(\alpha^\nabla, \beta^\nabla) := -\int_\square \mathrm{Pf}(\hat{\nabla}). \qquad \square$$

Notation. For two metric connections ∇^0 and ∇^1 on E we denote by $\mathrm{TPf}(\nabla^0, \nabla^1)$ the transgression form induced by the affine path $(1-s)\nabla^0 + s\nabla^1$.

If ∇^0 is obtained from ∇^1 through a section $s : M \to E$ of norm 1 by using the splitting

$$E = \mathbb{R}s \oplus \langle s \rangle^\perp \tag{2.5}$$

with $\nabla^0 := d \oplus P\nabla^1 P$, P being the orthogonal projection on $\langle s \rangle^\perp$ then we set $\mathrm{TPf}(\nabla^1, s) := \mathrm{TPf}(\nabla^0, \nabla^1)$. We will use the same notation even if s is only defined along a submanifold B

(or boundary) of M with the understanding that the splitting (2.5) holds only along B, ∇^0 is a connection on $E\big|_B \to B$ and consequently TPf is a form on B.

If s is clear from the context, we use $\mathrm{TPf}(\nabla^1)$ for $\mathrm{TPf}(\nabla^1, s)$. If the connection ∇^1 is the Levi-Civita connection of a metric g, then we use TPf^g for $\mathrm{TPf}(\nabla^1)$, like in Example 2.1.

Proposition 2.4. *For any 4 metric connections ∇^i, $0 \leq i \leq 3$, there exists a form γ such that*

$$\mathrm{TPf}(\nabla^0, \nabla^1) + \mathrm{TPf}(\nabla^1, \nabla^2) + \mathrm{TPf}(\nabla^2, \nabla^3) + \mathrm{TPf}(\nabla^3, \nabla^0) = d\gamma.$$

Proof. Put ∇^i in cyclic order at the vertices of a smooth map $\theta : \square \to \mathcal{A}$ which on the edges of $\square$ gives the affine path connecting ∇^i and ∇^{i+1}. The proof goes on as in Proposition 2.3. ☐

Proposition 2.5. *Let M be a Riemannian manifold (with or without boundary). Let ∇^0 and ∇^1 be two metric connections and $s : M \to E$ a smooth section of norm 1. Then there exists a degree $2k-2$ form γ such that the following equality of pairs holds:*

$$(\mathrm{Pf}(\nabla^1), -\mathrm{TPf}(\nabla^1, s)) - (\mathrm{Pf}(\nabla^0), -\mathrm{TPf}(\nabla^0, s)) = (-d\mathrm{TPf}(\nabla^1, \nabla^0), \mathrm{TPf}(\nabla^1, \nabla^0) + d\gamma).$$

If s is only defined along a submanifold (or boundary) B of M then the same relation holds with the second components restricted to B.

Proof. The equality in the first component is clear by (2.2) and Remark 2.2.

For the second component, let $\nabla^{0c} := d \oplus P\nabla^0 P$ and $\nabla^{1c} := d \oplus P\nabla^1 P$, where P is the projection onto $\langle s \rangle^\perp$. Apply Proposition 2.4 to the connections $\nabla^{0c}, \nabla^0, \nabla^1, \nabla^{1c}$ to get:

$$\mathrm{TPf}(\nabla^0, s) - \mathrm{TPf}(\nabla^1, s) + \mathrm{TPf}(\nabla^{1c}, \nabla^{0c}) = -\mathrm{TPf}(\nabla^0, \nabla^1) + d\gamma = \mathrm{TPf}(\nabla^1, \nabla^0) + d\gamma.$$

But $\mathrm{TPf}(\nabla^{1c}, \nabla^{0c}) = 0$ because s is simultaneously parallel for ∇^{0c} and ∇^{1c} hence $\mathrm{Pf}(\tilde{\nabla})$ vanishes on the affine segment of connections from ∇^{0c} to ∇^{1c}. ☐

Remark 2.6. Proposition 2.5 has topological content. Suppose that s is a unit section of $E\big|_{\partial M}$. Each pair $(\mathrm{Pf}(\nabla^i), -\mathrm{TPf}(\nabla^i, s))$ is closed in $\Omega^{2k}(M, \partial M) := \Omega^{2k}(M) \oplus \Omega^{2k-1}(\partial M)$ for the differential

$$d(\omega, \gamma) := (-d\omega, \iota^*\omega + d\gamma).$$

Proposition 2.5 says that two such pairs determine the same relative cohomology class. In the compact case, this was proved in [10] by showing that such a pair is Lefschetz dual to the zero locus of a generic extension of s to M. In the classical case, when s is the unit normal of ∂M this is also a consequence of Chern-Gauss-Bonnet [8] since the map:

$$(\omega, \gamma) \to \int_M \omega + \int_{\partial M} \gamma$$

gives an isomorphism $H^{\dim M}(M, \partial M) \simeq \mathbb{R}$.

Proposition 2.7. *Let (M, g) be a manifold and let $\pi : E \to M$ be a Euclidean vector bundle with metric connection ∇ and sections $s_0, s_1 : M \to S(E)$. Suppose there exists a homotopy $(s_t)_{t \in [0,1]} : M \to S(E)$ between the two sections. Then there exists a smooth form η such that:*

$$\mathrm{TPf}(\nabla, s_1) - \mathrm{TPf}(\nabla, s_0) = d\eta.$$

Proof. Let τ be the tautological section of $\pi^*E \to S(E)$. The corresponding "tautological" transgression $\mathrm{TPf}(\pi^*\nabla, \tau) \in \Omega^*(S(E))$ satisfies:

$$(s_t)^*\mathrm{TPf}(\pi^*\nabla, \tau) = \mathrm{TPf}(\nabla, s_t), \qquad \forall\ t \in [0,1];$$
$$d\mathrm{TPf}(\pi^*\nabla, \tau) = \pi^*\mathrm{Pf}(\nabla).$$

The homotopy formula for the homotopy $H := (s_t)_{t \in [0,1]} : [0,1] \times M \to S(E)$ and $\omega = \mathrm{TPf}(\pi^*\nabla, \tau)$ implies that

$$\mathrm{TPf}(\nabla, s_1) - \mathrm{TPf}(\nabla, s_0) = d\int_{[0,1]} H^*\omega + \int_{[0,1]} dH^*\omega.$$

But $dH^*\omega = \pi_2^*\mathrm{Pf}(\nabla)$ where $\pi_2 : [0,1]\times M \to M$ is the projection. The fiber integral over the fibers of π_2 of any form of type $\pi_2^*\eta$ is zero. $\square$

Proposition 2.7 implies the following refinement of Proposition 2.5:

Proposition 2.8. *Let M be a manifold with or without boundary, let ∇^0 and ∇^1 be two metric connections on the Euclidean vector bundle E and $(s_t)_{t\in[0,1]} : M \to S(E)$ a smooth homotopy. Then there exists a degree $2k-2$ form γ such that:*

$$(\mathrm{Pf}(\nabla^1), -\mathrm{TPf}(\nabla^1, s_1)) - (\mathrm{Pf}(\nabla^0), -\mathrm{TPf}(\nabla^0, s_0)) = (-d\mathrm{TPf}(\nabla^1,\nabla^0), \mathrm{TPf}(\nabla^1,\nabla^0) + d\gamma).$$

If the homotopy is defined only along a submanifold (or boundary) B then the second components are defined only over B.

2.2. **Transgressions and metrics.** On an Euclidean vector bundle V of rank $2k$, it is convenient to identify the space of skew-symmetric endomorphisms $\mathrm{End}^-(V)$ with Λ^2V^* by the rule:

$$\mathrm{End}^-(V) \ni A \mapsto a_A(v,w) := \langle v, Aw\rangle = -\langle Av, w\rangle.$$

Notice that on $\mathbb{R}^2$, $\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}$ goes to $e_1^*\wedge e_2^*$. The Pfaffian of A is defined by

$$\mathrm{Pf}(A) = \tfrac{1}{k!}\langle a_A^{\wedge k}, \mathrm{vol}_{V^*}\rangle \in \mathbb{R}.$$

In an orthonormal basis of V, Pf is a polynomial with integral coefficients in the entries of A.

Clearly this definition can be extended to endomorphisms $A \in \mathcal{A}\otimes\mathrm{End}^-(V)$ with values in any algebra $\mathcal{A}$, with the inner product acting only on the Λ^*V component. Then $\mathrm{Pf}(A)\in\mathcal{A}$. In this note, $\mathcal{A}$ will be the algebra of differential forms on a manifold.

If ∇ is a metric connection on a Euclidean vector bundle E of rank $2k$, from the curvature tensor $F(\nabla)\in\Gamma(\Lambda^2T^*M\otimes\mathrm{End}^-(E))$ we get a form of degree 2 with values in Λ^2E^* called the curvature form and denoted here by the same symbol. Explicitly:

$$F(\nabla) : \Lambda^2TM\otimes\Lambda^2E \to \mathbb{R}, \qquad F(\nabla)(X,Y;Z,W) = \left\langle Z, \left([\nabla_X,\nabla_Y] - \nabla_{[X,Y]}\right)W\right\rangle.$$

Then $F(\nabla)^k \in \Lambda^{2k}T^*M\otimes\Lambda^{2k}E^*$, and $\mathrm{Pf}(F(\nabla))\in\Omega^{2k}(M)$. This definition agrees with (2.1). The operation of contraction with the volume element in the second component is sometimes called Berezin integral. Double forms, i.e., sections of $\Lambda^*T^*M\otimes\Lambda^*E^*$, form an algebra.

From now on we take $E = TM$. Let g_0, g_1 be two Riemannian metrics on M, and $\nabla^{g_0},\nabla^{g_1}$ the corresponding Levi-Civita connections. We want to find an explicit primitive of the difference $\mathrm{Pf}(R^{g_1}) - \mathrm{Pf}(R^{g_0})$. Set $g_s = (1-s)g + sg_1$, a 1-parameter family of Riemannian metrics on M, and define a Riemannian metric on $X := [0,1]\times M$ as a generalized cylinder [4]:

$$G = ds^2 + g_s.$$

It is easy to see that for every $x\in M$, the intervals $[0,1]\times\{x\}$ are geodesics in X. Therefore, parallel transport on X along these intervals preserves the orthogonal complement to ∂_s, i.e., TM. We get for each s a vector bundle isometry

$$\tau_s : (TM, g_0) \to (TM, g_s).$$

We identify in this way for all s the Euclidean vector bundles with metric connections (TM, g_s, ∇^{g_s}) with (TM, g_0, ∇^s), where $\nabla^s = \tau_s^{-1}\nabla^{g_s}\tau_s$. Clearly such an identification preserves the Pfaffian of the curvature:

$$\mathrm{Pf}(R^{g_s}) = \mathrm{Pf}(R^s),$$

where $R^s = F(\nabla^s)$ is the curvature of ∇^s. Write

$$\mathrm{Pf}(R^{g_1}) = \mathrm{Pf}(R^{g_0}) + \int_0^1 \frac{d}{ds}\mathrm{Pf}(R^{g_s})ds = \mathrm{Pf}(R^{g_0}) + \int_0^1\frac{d}{ds}\mathrm{Pf}(R^s)ds.$$

The advantage of the second expression over the first is that now we work in a fixed Euclidean vector bundle (TM, g_0) endowed with a family in s of metric connections ∇^s, and the coefficients of the Pfaffian polynomial depend on the metric but not on the connection. We compute

$$\partial_s \mathrm{Pf}(R^s) \otimes \mathrm{vol}_{g_0} = \frac{1}{k!}\partial_s\left((R^s)^k\right) = \frac{1}{(k-1)!}\dot{R}^s \wedge \left((R^s)^{k-1}\right).$$

It is well-known that $\dot{R}^s$ is d^{∇^s}-exact: indeed, let u, v be vector fields on X tangent to M and parallel in the ∂_s direction. For every vector field Y on M constant in s (i.e., $[\partial_s, Y] = 0$), write

$$\langle \nabla^s_Y u, v\rangle = \langle \nabla^0_Y u, v\rangle + \langle \theta^s(Y)u, v\rangle.$$

Then $\dot{\nabla}^s = \dot{\theta}^s$ and so $\dot{R}^s = d^{\nabla^s}\dot{\theta}^s$. From the second Bianchi identity, $d^{\nabla^s}R^s = 0$, so

$$\dot{R}^s \wedge (R^s)^{k-1} = d^{\nabla^s}\left(\dot{\theta} \wedge (R^s)^{k-1}\right).$$

For every double form $\mu \in \Lambda^*M \otimes \Lambda^{2k}M$, write $\mu = \mathcal{B}_{g_0}\mu \otimes \mathrm{vol}_{g_0}$, where $\mathcal{B}_{g_0}$ is the Berezin integral with respect to g_0. Since vol_{g_0} is parallel, we have $d^{\nabla^s}\mu = d(\mathcal{B}_{g_0}\mu) \otimes \mathrm{vol}_{g_0}$. Hence

$$\frac{\partial}{\partial s}\mathrm{Pf}(R^s) = \frac{1}{(k-1)!}d\left(\mathcal{B}_{g_0}\left(\dot{\theta}^s \wedge (R^s)^{k-1}\right)\right).$$

It follows that

$$\mathrm{Pf}(R^{g_1}) = \mathrm{Pf}(R^{g_0}) + \frac{1}{(k-1)!}d\left(\int_0^1 \mathcal{B}_{g_0}(\dot{\theta}^s \wedge (R^s)^{k-1})\right). \tag{2.6}$$

Proposition 2.9. *Let $\alpha^\nabla(s) := \nabla^s$ be the above family of g_0-compatible connections. Then*

$$\frac{1}{(k-1)!}\int_0^1 \mathcal{B}_{g_0}(\dot{\theta}^s \wedge (R^s)^{k-1}) = \mathrm{TPf}(\alpha^\nabla).$$

Proof. Let $\tilde{\nabla} := \frac{d}{ds} + \nabla^s$ be the connection on π_2^*TM used in the previous subsection. By definition $\mathrm{TPf}(\alpha^\nabla) = \int_{[0,1]} \mathrm{Pf}(\tilde{\nabla})$, where the integration is over the fibers of $\pi_2 : X \to M$.

Every form γ on X is a sum of type $ds \wedge \omega_s + \eta_s$ where $(\omega_s)_{s\in[0,1]}$ and $(\eta_s)_{s\in[0,1]}$ are smooth families of smooth forms on M. Fiber integration kills the component η_s which does not contain the volume form of the fiber. In other words:

$$\int_{[0,1]} \gamma = \int_{[0,1]} ds \wedge \omega_s = \int_0^1 \omega_s \, ds = \int_0^1 \iota_{\partial_s}\gamma \, ds.$$

The integrals $\int_0^1 (\cdot)\, ds$ is to be understood as integrals of functions (of s) with values in Λ^*T_pM for $p \in M$. We need compute $\iota_{\partial_s}(\mathrm{Pf}(\tilde{\nabla}))$. First notice that $F(\tilde{\nabla}) = ds \wedge \dot{\nabla}^s + F(\nabla^s)$. Then

$$F(\tilde{\nabla})^k = k\, ds \wedge \dot{\nabla}^s \wedge F(\nabla^s)^{k-1} + F(\nabla^s)^k \tag{2.7}$$

The contraction operation $\iota_{(\cdot)}$ can be defined equally well on forms with values in an algebra. Then

$$\iota_{\partial_s}(\mathrm{Pf}(\tilde{\nabla})) = \frac{1}{k!}\iota_{\partial_s}(\mathcal{B}_{g_0}(F(\tilde{\nabla})^k)) = \frac{1}{k!}\mathcal{B}_{g_0}(\iota_{\partial_s}[F(\tilde{\nabla})^k])$$

where ι_{∂_s} acts by definition only on the first component of a double form[1]. The second equality holds because $\mathcal{B}_{g_0}$ acts on the second component only of the double form. By (2.7),

$$\iota_{\partial_s}(\mathrm{Pf}(\tilde{\nabla})) = \frac{1}{(k-1)!}\mathcal{B}_{g_0}(\dot{\theta}^s \wedge F(\nabla^s)^{k-1}). \qquad \square$$

---

[1] The curvature form is in general a section of $\Lambda^2T^*M \otimes \Lambda^2E^*$.

3. GAUSS-BONNET ON MANIFOLDS WITH BOUNDARY

This section contains a new proof of the well-known generalization of Gauss-Bonnet on manifolds with boundary. It is based on the formalism of double-forms used in the previous section and sets the stage for the computations in the singular case.

Let g be a smooth metric on a compact manifold M^{2k} with boundary ∂M. Let $R^h \in \Lambda^2\partial M \otimes \Lambda^2\partial M$ be the curvature form of the boundary with respect to the induced metric h and II the second fundamental form of $\partial M \hookrightarrow M$. Our convention here is the following:

$$\mathrm{II}(X,Y) = -\langle \nabla_X \nu, Y\rangle$$

where ν is the exterior unit normal. We will use the symbol II also for the $(1,1)$ double form on ∂M determined by II. We denote by Pf^g the Pfaffian of g and by TPf^g the transgression form on ∂M constructed from ∇^g and $d\oplus\nabla^h$ (see Example 2.1) where ∇^g and ∇^h are the Levi-Civita connections on M and ∂M respectively. We give a direct proof of the Allendoerfer-Weil-Gauss-Bonnet-Chern [9] formula for manifolds with boundary using the formalism of double forms.

Proof of the Gauss-Bonnet-Chern formula (1.1). Let $g_1 := g$. Using the unit geodesic flow normal to the boundary, we can write (M,g) as a generalized cylinder [4] near the boundary:

$$g = dt^2 + h(t),$$

where $h(t)$ is a smooth family of symmetric 2-tensors on ∂M, and $h(0)$ is a metric. Take g_0 to be any metric which in the same product decomposition near the boundary looks like

$$g_0 = dt^2 + h(0),$$

i.e., g_0 is of product type near the boundary and induces the same metric $h(0)$ on ∂M as g. By the Gauss-Bonnet formula for product-type metrics (obtained by doubling the manifold for example) and the transgression formula (2.6), we get

$$(2\pi)^k\chi(M) = \int_M \mathrm{Pf}(R^{g_0}) = \int_M \mathrm{Pf}(R^g) - \frac{1}{(k-1)!}\int_0^1\int_{\partial M} \mathcal{B}_{g_0}\left(\dot\theta^s \wedge (R^s)^{k-1}\right). \tag{3.1}$$

Notice that all metrics g_s coincide on $TM\big|_{\partial M}$. One consequence is that all bundle isometries τ_s when restricted to $TM\big|_{\partial M}$ are equal to the identity. Hence every Levi-Civita connection ∇^{g_s} when restricted to $TM\big|_{\partial M}$ is equal to ∇^s and all are metric compatible whether we refer to g_0 or g. By Propostion 2.9 the integral on the boundary in (3.1) is in fact equal to $\iota^*\mathrm{TPf}(\alpha^\nabla)$ where $\iota^* : \partial M \to M$ is the inclusion and $\alpha^\nabla(s) = \nabla^s$. By Proposition 2.3 when integrating over the boundary, it does not matter what path of connections one takes between the first and the last connection so we might as well take the segment. To complete the proof of (1.1) we still have to identify explicitly the transgression term from (3.1).

First, the Berezin integrals with respect to g and to h at the boundary are related by

$$\mathcal{B}_g((1\otimes dt)\wedge\mu) = \mathcal{B}_h(\mu)$$

for every form $\mu \in \Lambda^{2k-1}\partial M$.

We look at the restriction $\theta^s : T\partial M \to \mathrm{End}^-(TM\big|_{\partial M})$. We claim that as $(1,2)$ forms:

$$\theta^s = (1\otimes dt)\wedge s\mathrm{II}^g. \tag{3.2}$$

By definition $\theta^s = \nabla^{g_s} - \nabla^{g_0}$ and $\langle \nabla^{g_s}_X Y, Z\rangle = \langle \nabla^g_X Y, Z\rangle$ for all $X,Y,Z \in T\partial M$ as $g_s \equiv h$ on $T\partial M$. Moreover $\langle \nabla^{g_s}_X \partial_t, \partial_t\rangle = 0$ for all s and $X \in T\partial M$. Hence with respect to the decomposition $TM\big|_{\partial M} = \mathbb{R}\partial_t \oplus T\partial M$ and the corresponding decomposition of $\mathrm{End}^-(TM\big|_{\partial M})$, the only non-zero components of θ^s are off-diagonal. Then for $X,Y\in T\partial M$

$$\begin{aligned}\langle \theta^s_X(Y), \partial_t\rangle = \mathrm{II}^{g_s}(X,Y) = -\langle \nabla^{g_s}_X\partial_t, Y\rangle = -\frac{1}{2}(L_{\partial_t}g_s)(X,Y)\\ = -\frac{s}{2}h'(0)(X,Y) = -\frac{s}{2}L_{\partial_t}g(X,Y) = s\mathrm{II}^g(X,Y).\end{aligned} \tag{3.3}$$

where we used Lemma 3.1 in the first line. Notice that (3.3) is a rewritting of (3.2).

Since $\nabla^s = \nabla^0 + \theta^s$ we get that $R^s = R^0 + d^{\nabla^0}\theta^s + \theta^s \circ \theta^s$ where we use the symbol $\circ$ instead of the more popular $\wedge$ in order to distuinguish it from the product for double forms.

On one hand, $R^0 = 0 \oplus R^h$ with respect to $TM\big|_{\partial M} = \mathbb{R}\partial_t \oplus T\partial M$. Hence as $(2,2)$ forms on ∂M one has $R^0 = R^h$. Second, d^{∇^0} also respects this decomposition so $d^{\nabla^0}\theta^s$ will be a two form with non-zero values only on the anti-diagonal blocks of $\operatorname{End}^-(TM\big|_{\partial M})$. It follows that when writing $d^{\nabla^0}\theta^s$ as a double form, the second component will always contain a dt. But $\dot{\theta}^s$ also contains a dt in its second component. So in $\dot{\theta}^s \wedge (R^s)^{k-1}$ this product vanishes.

We are left with turning $\theta^s \circ \theta^s$ into a double form. If $\{\partial_t, e_2, \ldots, e_n\}$ is an oriented orthonormal basis for TM at a point $p \in \partial M$ then at p, θ^s is a skew-symmetric matrix with non-zero terms only along the first line and the first column. In fact $\theta^s_{1i} = s\mathrm{II}^g(\cdot, e_i)$, $i \geq 2$ and

$$(\theta^s \circ \theta^s)_{ij} = -s^2 \mathrm{II}^g(\cdot, e_i) \wedge \mathrm{II}^g(\cdot, e_j), \qquad\qquad i < j.$$

This represents the $(2,2)$ double form

$$-s^2 \sum_{2 \leq i < j} \mathrm{II}^g(\cdot, e_i) \wedge \mathrm{II}^g(\cdot, e_j) \otimes e_i^* \wedge e_j^*.$$

On the other hand

$$\mathrm{II}^g \wedge \mathrm{II}^g = \left(\sum_{i \geq 2} \mathrm{II}^g(\cdot, e_i) \otimes e_i^*\right) \wedge \left(\sum_{i \geq 2} \mathrm{II}^g(\cdot, e_i) \otimes e_i^*\right) = 2 \sum_{2 \leq i < j} \mathrm{II}^g(\cdot, e_i) \wedge \mathrm{II}^g(\cdot, e_j) \otimes e_i^* \wedge e_j^*.$$

Hence $\theta^s \circ \theta^s = -\frac{s^2}{2}\mathrm{II}^g \wedge \mathrm{II}^g$, and so the integrand over ∂M in (3.1) is

$$\frac{1}{(k-1)!}\int_0^1 \mathcal{B}_g\left((1 \otimes dt) \wedge \mathrm{II}^g \wedge \left(R^h - \frac{s^2}{2}(\mathrm{II}^g)^2\right)^{k-1}\right) ds =$$

$$= \frac{1}{(k-1)!}\mathcal{B}_h\left(\sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{2^j} \frac{1}{2j+1} (\mathrm{II}^g)^{2j+1} \wedge (R^h)^{k-1-j}\right). \qquad \square$$

The next simple Lemma is quite well-known and is widely used in this article.

Lemma 3.1. *Let TM be endowed with a metric G and corresponding Levi-Civita connection ∇. Let $X \in \Gamma(TM)$ be a vector field such that $X^\sharp$ is a closed 1-form (e.g. if X is gradient). Then*

$$G(\nabla_Y X, Z) = \tfrac{1}{2}(L_X G)(Y, Z).$$

Proof. Directly from the Koszul formula one has

$$2G(\nabla_Y X, Z) = (L_X G)(Y, Z) + dX^\sharp(Y, Z).$$

By hypothesis the second term vanishes. $\square$

Remark 3.2. Not only that the integral over ∂M of TPf^g equals the integral on ∂M of the right hand side of (3.1) but the integrands themselves coincide. This is because the Levi-Civita connection for $g_s = (1-s)g_0 + sg$, when restricted to $TM\big|_{\partial M}$ coincides with $(1-s)\nabla^{g_0} + s\nabla^g$. This follows from $g_s \equiv g_0$ on $TM\big|_{\partial M}$ for all s and from the Koszul formula which always gives:

$$\langle \nabla^{g_s}_X Y, Z\rangle_{g_s} = (1-s)\langle \nabla^{g_0}_X Y, Z\rangle_{g_0} + s\langle \nabla^{g_1}_X Y, Z\rangle_g.$$

Remark 3.3. Let

$$\frac{\mathcal{B}_h\left((R^h)^j \wedge \mathrm{II}^{2k-1-2j}\right)}{j!(2k-1-2j)!} =: \mathcal{G}^h_{j,2k-1}.$$

Then the integral of the transgression form has the following aesthetically pleasing form

$$\sum_{j=0}^{k-1}(-1)^j(2j-1)!!\int_{\partial M}\mathcal{G}^h_{k-1-j,2k-1}.$$

Example 3.4. The Gauss-Bonnet formula 1.1 applied to the unit disk $D^{2n}\subset\mathbb{R}^{2n}$ anticipates that

$$\frac{1}{(2\pi)^k}\int_{S^{2n-1}}\mathrm{TPf}^g=-1.$$

The sphere is oriented with the outer normal first convention. We compute the right hand side of (1.1) to check this. On one hand, $\mathrm{II}=-h$, where h is the round metric. On the other hand, Gauss equation gives $0=R^h-\frac{1}{2}\mathrm{II}\wedge\mathrm{II}$, hence

$$\mathcal{B}_h((R^h)^j\wedge\mathrm{II}^{2k-2j-1})=-\frac{1}{2^j}\mathcal{B}_h(h^{2k-1})=-\frac{1}{2^j}(2k-1)!\mathrm{vol}_h.$$

Using that $\mathrm{vol}(S^{2k-1})=\frac{2\pi^k}{(k-1)!}$ we get

$$\begin{aligned}&\sum_{j=0}^{k-1}c(j,k)\int_{S^{2n-1}}\mathcal{B}_h((R^h)^j\wedge\mathrm{II}^{2k-2j-1})\\ &=-\sum_{j=0}^{k-1}\frac{(-1)^{k-1-j}}{2^{k-1-j}}\frac{1}{j!}\frac{1}{(k-1-j)!}\frac{1}{2k-2j-1}\frac{(2k-1)!}{2^j}\mathrm{vol}(S^{2k-1})\\ &=-\frac{(2k-1)!}{2^{k-1}}\frac{2\pi^k}{[(k-1)!]^2}\sum_{j=0}^{k-1}\frac{(-1)^j\binom{k-1}{j}}{2j+1}.\end{aligned}$$

Notice that

$$\sum_{j=0}^{k-1}\frac{(-1)^j\binom{k-1}{j}}{2j+1}=\int_0^1(1-x^2)^{k-1}\,dx=\int_0^{\pi/2}(\cos\theta)^{2k-1}\,d\theta=\frac{2^{2k-2}[(k-1)!]^2}{(2k-1)!}.$$

Hence

$$\frac{1}{(2\pi)^k}\sum_{j=0}^{k-1}c(j,k)\int_{S^{2n-1}}\mathcal{B}_h((R^h)^j\wedge\mathrm{II}^{2k-2j-1})=-1.$$

Remark 3.5. The integrand in (1.1) on ∂M coincides with Chern's integrand [8]. Chern's transgression, which lives on the spherical bundle SM, can be written (see for example [28]) as[2]

$$\Pi:=-\sum_{j=0}^{k-1}a_iA_i,\qquad a_i=[(2\pi)^k i!(2k-2i-1)!!]^{-1},\qquad A_i=(\pi^*\mathcal{R})^i\wedge I\wedge(DI)^{2k-2i-1}.\qquad(3.4)$$

In (3.4), $\mathcal{R}$ is the curvature form on M, $I:SM\to\pi^*TM$ is the tautological section seen as a 0-form on SM with values in π^*TM and $DI=(\pi^*\nabla)I$ is the covariant derivative seen as a 1-form with values in π^*TM. Hence one works in the algebra of forms on SM with values in $\Lambda^*\pi^*TM$. Wedging with I kills the normal component in any product $DI^{2k-2h-1}$ and also in $(\pi^*\mathcal{R})^i$.

Given a hypersurface N oriented by the normal ν one has that $\nu^*(I\wedge DI)=\nu\wedge\nu^*(DI)$ actually equals $-\nu\wedge\mathrm{II_N}$ where $\mathrm{II_N}:\mathrm{TN}\to\mathrm{TN}$ is the second fundamental form seen as the endomorphism $-\nabla\nu$. Moreover $\nu^*\mathcal{R}$ is the tangential component of the curvature tensor of M restricted to N. Let $\mathrm{II}:=\mathrm{II}_N$ and R^N the curvature form on N. Gauss Equation gives

$$\nu^*\mathcal{R}=R^N-\frac{1}{2}\mathrm{II}\wedge\mathrm{II}.$$

---

[2] The negative sign in front of the sum is there so that $d\Pi=\pi^*\mathrm{Pf}^g$.

Therefore

$$-\nu^*(A_i) = (R^N - 1/2\mathrm{II} \wedge \mathrm{II})^i \wedge \nu \wedge \mathrm{II}^{\mathrm{2k-2i-1}}$$

and we must check that

$$\begin{aligned}
\nu^*\Pi &= \sum_{i=0}^{k-1} \sum_{j=0}^{i} \frac{1}{i! \cdot 1 \cdot 3 \cdot \ldots \cdot (2k-2i-1)} \frac{(-1)^j \binom{i}{j}}{2^j} \mathrm{II}^{2k-2(i-j)-1} (R^N)^{i-j} \\
&= \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}}{(k-1)!} \frac{1}{2j+1} \frac{(-1)^j}{2^j} \mathrm{II}^{2j+1} (R^N)^{k-1-j} =: \mathrm{TPf}^N,
\end{aligned}$$

This equality follows from the elementary identity of double factorials

$$\sum_{j=0}^{p} (-1)^j \frac{(2p)!!}{(2j)!!(2p-2j+1)!!} = \frac{(-1)^p}{2p+1}.$$

4. Conical manifolds

Let N be a compact oriented manifold, possibly disconnected. A *conical singularity* modeled on N is a Riemannian metric on $(-\epsilon, 0) \times N$ of the form

$$g_c = dr^2 \oplus f^2(r) \cdot h(r)$$

where $h(r)$ is a smooth family of Riemannian metrics on N down to $r = 0$ and $f : (-\epsilon, 0] \to [0, \infty)$ is a function with the following properties

- (i) f is smooth on $(-\epsilon, 0)$;
- (ii) f vanishes only at 0;
- (iii) f is C^1 in 0.

Notice that $f'(0) \leq 0$.

Definition 4.1. When $h(r) \equiv h$ is constant and $f(r) = -\theta r$ with $\theta > 0$ we call the conical singularity a *geometric cone* of inclination θ.

The smoothness at $r = 0$ of $h(r)$ needs to be emphasized. There are two equivalent formulations for this property:

1. The metric $dr^2 \oplus h(r)$ is the restriction to $(-\epsilon, 0) \times N$ of a smooth metric on $(-\epsilon, \epsilon) \times N$;
2. The family $(-\epsilon, 0) \ni r \mapsto h(r) \in C^\infty(N, T^*N \otimes T^*N)$ has a limit at $r = 0$ together with all its derivatives in r.

Definition 4.2. An oriented manifold with conical-type singularities is a Riemannian manifold (M, g) such that there exists a compact set K and an orientation preserving diffeomorphism $\varphi : M \setminus K \simeq (-\epsilon, 0) \times N$ such that on $M \setminus K$:

$$g = \varphi^* g_c.$$

We now define some polynomials in the curvature of a Riemannian manifold (N, h) of dimension n using the Berezin integral $\mathcal{B}_h$ where

$$h := h(0).$$

Definition 4.3. The Lipschitz-Killing curvature (see [17] or [21]) of level j is, up to a normalization constant, the following form of degree n on N:

$$P_{j,n}(h) = \frac{1}{j!(n-2j)!} \mathcal{B}_h \left((R^h)^j \wedge h^{n-2j} \right).$$

Like the Pfaffian, in any orthonormal base the form $P_{j,n}$ is a polynomial with integral coefficients in the components of R^h. The Lipschitz-Killing curvatures are familiar objects and they appear in Weyl's tube formula.

Example 4.4. Here are a few examples:

$$P_{0,n}(h) = \mathrm{vol}_h, \qquad P_{1,n}(h) = \frac{1}{2}\mathrm{scal}_h \cdot \mathrm{vol}_h, \qquad P_{k,2k} = \mathrm{Pf}(R^h).$$

Remark 4.5. Let $\tilde{N} := (-\epsilon, 0) \times N$ be a geometric cone of inclination $c > 0$. Then the transgression form for each slice $\{r\} \times N$ does not depend on r. Indeed the Levi-Civita connection and the cylindrical connection obtained from it are the same for $T\tilde{N}|_{\{r\}\times N}$ irrespective of r. Denote this transgression form by $\mathrm{TPf}(N, h, c)$. For the inclination $c = 0$, set $\mathrm{TPf}(N, h, c) = 0$.

We prove now the main result of this section.

Theorem 4.6. *Let (M^{2k}, g) be an oriented manifold with conical-type singularities modeled on a possibly disconnected manifold N with induced metric h. Then*

$$(2\pi)^k \chi(M) = \int_M \mathrm{Pf}^g - \int_N \mathrm{TPf}(N, h, -f'(0)) \tag{4.1}$$

$$= \int_M \mathrm{Pf}^g + \sum_{j=0}^{k-1} [f'(0)]^{2k-2j-1} \tilde{c}(k-1-j) \int_N P_{j,2k-1}(h) \tag{4.2}$$

with

$$\tilde{c}(l) = (-1)^l \cdot (2l-1)!!. \tag{4.3}$$

Proof. For each $r \in (-\epsilon, 0)$, let M_r be the complement of $\varphi^{-1}((r, 0) \times N)$. It is a compact manifold with boundary and therefore (1.1) applies to it:

$$(2\pi)^k \chi(M_r) = \int_{M_r} \mathrm{Pf}^g - \int_{\partial M_r} \mathrm{TPf}^g.$$

Clearly all M_r are homotopic to each other so the left hand side does not change with r. We will show that

$$\lim_{r\to 0} \int_{\partial M_r} \mathrm{TPf}^g = -\sum_{j=0}^{k-1} [f'(0)]^{2k-1-2j} \tilde{c}(k-1-j) \int_N P_{j,2k-1}(h).$$

This will also prove the convergence of $\int_{M_r} \mathrm{Pf}^g$ when $r \to 0$. (In Section 7 we prove the stronger statement that Pf^g is a smooth form on M.)

The first observation is that the Levi-Civita connection ∂M_r with the metric $h_1(r) := f(r)^2 h(r)$ is the same as the Levi-Civita connection for the metric $h(r)$, hence as operators

$$R^{h_1(r)} = f(r) R^{h(r)}$$

due to the metric dependence of the identification $\mathrm{End}^-(V) \simeq \Lambda^2 V^*$.

One is left computing the evolution of II^r for ∂M_r. Since ∂_r is a gradient vector field we apply Lemma 3.1 again:

$$\mathrm{II}^r(X, Y) = -\langle \nabla^g_X \partial_r, Y\rangle = -\frac{1}{2} L_{\partial r}(dr^2 + f^2(r)h(r)) = -[f'(r)f(r)h(r) + \frac{f^2(r)}{2} h'(r)].$$

We also have $\mathcal{B}_{h_1(r)} = (f(r))^{1-2k} \mathcal{B}_{h(r)}$ and so

$$\mathcal{B}_{h_1(r)}\left((R^{h_1(r)})^j \wedge (\mathrm{II}^r)^{2k-1-2j}\right) = -f'(r)^{2k-1-2j} \mathcal{B}_{h(r)}\left((R^{h(r)})^j \wedge h(r)^{2k-1-2j}\right) + o(f(r)).$$

Multiply this with $c(j,k) = \frac{\tilde{c}(k-1-j)}{j!(2k-1-2j)!}$, take the sum in j and the limit $r \to 0$ to get (4.2).

To see that (4.1) is true, recall (for example Remark 3.2) that $\int_N \mathrm{TPf}(N, h, c)$ can be computed also as a sum of integrals over N of products $\mathrm{II}^{2k-1-2j} \wedge R^j$ where II and R are the second fundamental form respectively the curvature form of a slice of a geometric cone. One notices that for a geometric cone II is a multiple of the metric and the computations go as before. □

We notice thus that for an odd-dimensional manifold the total Lipschitz-Killing curvatures can be recovered as coefficients of the integral of a certain transgression. We state this separately.

Corollary 4.7. *For a geometric cone modeled on* (N, h) *of inclination* θ *with* $\dim N = n$, n *odd, the following holds:*

$$\int_N \mathrm{TPf}(N, h, \theta) = \sum_{j=0}^{\frac{n-1}{2}} \theta^{n-2j} \tilde{c}\left(\frac{n-1}{2} - j\right) \int_N P_{j,n}(h).$$

Proof. The function is in this case $f(r) = -\theta r$. □

Remark 4.8. What we called odd Pfaffian in the Introduction is in fact

$$\mathrm{Pf}^{\mathrm{odd}}(h) = \mathrm{TPf}(N, h, 1).$$

Notice that in the case when $N = S^{2n-1}$ with the round metric we get:

$$\int_{S^{2n-1}} \mathrm{TPf}(S^{2n-1}, \mathrm{round}, 1) = 1.$$

One can compare this with Example 3.4. The difference in sign has to do with the fact that S^{2n-1} seen as a geometric cone is oriented with the inner normal first since that is the direction of ∂_r that points towards the "singularity".

Remark 4.9. We can construct a manifold with boundary $\tilde{M} := M \cup (-\epsilon, 0] \times N/\sim$ where the identifcation is made via the diffeomorphism φ of Definition 4.2 in an obvious way. The degenerated conical metric g induces a pseudo-distance on $\tilde{M}$ in which the (pseudo) distance between any two points on $\partial\tilde{M}$ is zero. Collapsing the boundary of $\tilde{M}$ to a point gives a metric space $\hat{M}$ which is homeomorphic to the one point compactification of M. Then

$$\chi(\hat{M}) = 1 + \chi(M).$$

If the singular space $\hat{M}$ is the focus of the analysis, then we can say that the singularity, or the point at ∞ contributes to the Euler characteristic with the quantity

$$1 + \frac{1}{(2\pi)^k} \sum_{j=0}^{k-1} f'(0)^{2k-1-2j} \tilde{c}(k-1-j) \int_N P_{j,2k-1}(h).$$

Example 4.10. In the case $k = 1$ the contribution of the singularity is (recall that $f'(0) \leq 0$, $\tilde{c}(0,1) = 1$)

$$1 + \frac{f'(0)}{2\pi} \mathrm{length}_h(N).$$

This fits with two opposite examples. The first is a closed surface S embedded in $\mathbb{R}^3$ with a cuspidal singularity. Then $f'(0) = 0$. The geometric contribution to the Euler characteristic of the cusp is 1 which is the area of the half unit sphere divided by 2π. The half unit sphere is the normal cycle of the cusp, or the solid angle described by the variation of a unit normal to each surface of a family of smooth surfaces contained in the bounded region of S and converging to S.

The other example is when $N = S^1$ with the round metric and $f'(0) = -1$. Then $\hat{M}$ is a closed surface with smooth metric (see [23] Pag 13, Prop.1) and the contribution of the removable singularity vanishes, recovering Gauss-Bonnet for $\hat{M}$ in this case.

5. Edge manifolds: the model metrics

Let N be an n-dimensional closed, oriented manifold. Assume $\pi : N \to B$ is a locally trivial fiber bundle with vertical bundle VN and suppose π is endowed with an Ehresmann connection $\mathcal{E} \in \mathrm{Hom}(TN, VN)$ that induces a decomposition

$$TN = VN \oplus \pi^* TB.$$

An edge singularity modeled on $(N, \pi, \mathcal{E})$ is a metric on $(-\epsilon, 0) \times N$ of the type $dr^2 \oplus r^2 g^V \oplus \pi^* g^B$ where g^V and g^B are metrics on VN and TB respectively. More generally, a *model edge metric* will be any metric of type:

$$g_e = dr^2 \oplus r^2 g^V(r) \oplus \pi^* g^B$$

where $g^V(r)$ is a smooth family of metrics down to $r = 0$. We set:

$$g^V := g^V(0), \qquad\qquad g^N := g^V \oplus \pi^* g^B.$$

The Levi-Civita connection ∇^N of the metric g^N on N induces a connection ∇^{VN} on VN via $\mathcal{E}\nabla^N\mathcal{E}$. We will call it the orthogonal projection. Clearly ∇^{VN} restricted to each fiber N_b is the Levi-Civita connection of that fiber for the metric g^V.

Definition 5.1. A manifold with edge singularities is a smooth manifold M with a Riemannian metric g such that there exists a compact set K and a diffeomorphism $\varphi : M \setminus K \to (-\epsilon, 0) \times N$, such that on $M \setminus K$:

$$g = \varphi^* g_e.$$

Proof of Theorem 1.2. Let $b := \dim B$ and $f := 2k - 1 - b$ be the dimension of the fiber of π. As in the conical case, the Euler characteristic of M_r is constant and equal to $\chi(M)$. So it is enough to prove the convergence of the integrals of transgression forms in (1.1) for the slices $\partial M_r \simeq \{r\} \times N$.

We will use the following terminology for double forms of type $(2,2)$ on N. A form is called (purely) horizontal if its second component belongs to $\Gamma(\pi^*\Lambda^2 T^* B)$. It is called (purely) vertical its second component belongs to $\Gamma(\Lambda^2 V^* N)$. It is a mixed form if its second component belongs to $\Gamma(\pi^* T^* B \otimes V^* N \oplus V^* N \otimes \pi^* T^* B)$. Clearly every $(2,2)$ form can be written as a sum of a purely horizontal, a purely vertical and a mixed form.

The technical part of the proof is to decompose the curvature form of the slice $\{r\} \times N$ for the metric $g_r := r^2 g^V(r) \oplus \pi^* g^B$ into its horizontal, vertical and mixed components. This is the object of Proposition 5.10 below, according to which the curvature $F(\nabla^{g_r})$ for the slice $\{r\} \times N$ with metric $r^2 g^V(r) \oplus \pi^* g^B$ decomposes as follows.

$$F(\nabla^{g_r}) = (A_0 + A_2 r^2 + A_4 r^4) + r^2(C_2 + r^2 C_4) + r^2(D_2 + r^2 D_4) = X(r) + r^2 Y(r)$$

where $A_0, A_2, A_4, C_2, C_4, D_2, D_4$ are geometric quantities which depend smoothly on r down to $r = 0$, and are constant when g^V is constant in r. Moreover, for all i, A_i is purely horizontal, D_i is purely vertical and C_i is mixed. We have $A_0 = \pi^* F(\nabla^B)$ and $D_2 = F(\nabla_r^{VN})$ and this is all we need to know for computations. Then

$$X(r) := A_0 + A_2 r^2 + A_4 r^4, \qquad\qquad Y(r) := C_2 + D_2 + r^2(C_4 + D_4)$$

is a convenient separation of the terms.

On the other hand, applying Lemma 3.1 yet again we conclude that:

$$\mathrm{II}^r = -\left(r g^V(r) + \frac{r^2}{2}\dot{g}^V(r)\right) =: -rZ$$

where Z is a vertical $(1,1)$ double form

As for the Berezin integrals, one has (remember that r is negative):

$$\mathcal{B}_{g_r}(\cdot) = \frac{1}{(-r)^f}\mathcal{B}_{g^N}(\cdot).$$

Then

$$F(\nabla^{g_r})^j \wedge (\mathrm{II}^r)^{2k-1-2j} = \sum_{i=0}^{j} \binom{j}{i} \cdot (-r)^{2k-1-2i} X^i \wedge (Y^{j-i} Z^{2k-1-2j}).$$

Hence

$$\mathcal{B}_{g_r}\left(F(\nabla^{g_r})^j \wedge \mathrm{II}^{2k-1-2j}\right) = \sum_{i} \binom{j}{i} (-r)^{b-2i} \mathcal{B}_{g^N}\left(X^i \wedge (Y^{j-i} Z^{2k-1-2j})\right). \tag{5.1}$$

Notice that X^i is a purely horizontal double form of bi-degree $(2i, 2i)$, hence it vanishes if $2i > b$.

On the other hand for $2i < b$ all forms $\omega = \mathcal{B}_{g^N}(X^i \wedge (Y^{j-i} Z^{2k-1-2j}))$ have a limit when $r \to 0$. Therefore only the term $2i = b$ survives in the sum (5.1) when $r \to 0$.

Hence, if b is odd the limit is 0. If b is even we get

$$\binom{j}{b/2} \mathcal{B}_{g^N}\left(X(0)^{b/2} Y(0)^{j-b/2} Z(0)^{2k-1-2j}\right).$$

Now $Y(0) = C_2 + D_2$ and C_2 is a mixed term. Since $X(0)^{b/2}$ is a purely horizontal form of maximal bi-degree, it will kill all terms that contain a horizontal component in $Y(0)^{j-b/2}$. Hence only $D_2^{j-b/2}$ will survive. We are left with

$$\binom{j}{b/2} \mathcal{B}_{g^N}\left((\pi^* F(\nabla^B))^{b/2} F(\nabla^{VN})^{j-b/2} (g^V)^{2k-1-2j}\right).$$

Multiplying with $c(j,k)$, integrating and summing over $0 \le i := j - b/2 \le (f-1)/2$ gives the result, since $k - j - 1 = (f-1)/2 - i$. □

Example 5.2. Let $\pi : E \to B$ be a Euclidean vector bundle of rank $2k$ endowed with a metric connection ∇. Then $\pi^*\nabla$ and the tautological section τ determine on $SE := \{v \in E \mid |v| = 1\}$ a transgression form $\mathrm{TPf}(\pi^*\nabla, \tau)$ of degree $2k-1$ with the property:

$$\frac{1}{(2\pi)^k} \int_{SE/B} \mathrm{TPf}(\pi^*\nabla, \tau) \equiv 1 \tag{5.2}$$

when the fibers of $SE \to B$ are oriented via the interior normals. This reduces immediately to Example 3.4 (see also Remark 4.8).

Suppose now that B is a proper submanifold of a compact Riemannian manifold $\hat{M}$, both of even dimension. The normal bundle νB inherits a metric which is obviously a model edge metric with $N = S(\nu B)$. Assume for the moment that the normal exponential map induces an isometry $D_\epsilon(\nu B) \to U$ onto a neighborhood U of B where $D_\epsilon(\cdot)$ is the disk bundle of radius ϵ. Let $M := \hat{M} \setminus B$. Then using (5.2), Theorem 1.2 reduces to

$$\chi(M) = \chi(\hat{M}) - \chi(B).$$

Clearly this relation is also a topological consequence of Mayer-Vietoris for the cover $\{M, U\}$ of $\hat{M}$. The same identity holds when $\dim B$ is odd, albeit in that case $\chi(B) = 0$.

The results of Section 7 show that the hypothesis that the normal exponential map be an isometry is unnecessary.

Example 5.3. A more general situation when the integral of the transgression form is independent of the fiber is the following. Let $P \to B$ be a principal bundle with structure group G. Suppose G acts by isometries on a Riemannian manifold F. Let $N := P \times_G F$ be the associated fiber bundle over B. This is another way of saying that the fiber bundle with fiber F has transition maps taking values in $G \subset \mathrm{Isom}(F)$ Then the vertical bundle VN inherits a Riemannian metric since $VN \simeq P \times_G TF$ with G acting on TF via the differentials of the isometries. Since TF has a metric to start with and G preserves it then one will have a metric on VN.

Any G-principal connection $\omega \in \Omega^1(P;\mathfrak{g})$ gives rise to a parallel transport via isometries between the fibers of $N \to B$. Clearly the transgression form $\mathrm{TPf}(N_b, g^{N_b}, 1)$ of a fiber N_b obtained from the conical metric $dr^2 \oplus r^2 g^{N_b}$ on $(-1,1)\times N$ depends only on the isometry class of the metric g^{N_b}. Therefore in the situation when all the fibers are isometric, the integral will be constant.

Remark 5.4. One might ask what happens when $\dim M = 2k+1$ is odd with an edge singularity. If we look at M_r which is a compact manifold of odd dimension with boundary then by Lefschetz Duality one gets that $\chi(M_r) = \frac{1}{2}\chi(\partial M_r)$.

Now, $\chi(\partial M_r) = \int_{\partial M_r} \mathrm{Pf}(\nabla^{g_r})$ is constant with respect to r. If one uses as above the decomposition of $F(\nabla^{g_r})$ into its horizontal, mixed and vertical components then for B even dimensional one gets

$$\chi(N) = \lim_{r\to 0} \frac{1}{(2\pi)^k(2k)!}\int_{\partial M_r} F(\nabla^{g_r})^{2k} = \int_B \mathrm{Pf}(g^B)\int_{N/B}\mathrm{Pf}(g^V) = \chi(B)\chi(F)$$

while for odd $\dim B$ one gets zero. We recover thus a Riemannian-geometric proof of the multiplicativity of Euler characteristic in fibrations.

5.1. **The curvature form of a Riemannian submersion.** In order to completely describe the decomposition of the curvature form $F(\nabla^{g_r})$ into its vertical, horizontal and mixed components we need to introduce a few objects. Let $u = r^2$ and look at the adiabatic deformation of the metric on N:

$$h_u := g_u^V \oplus u^{-1}\pi^* g^B.$$

In this section we are interested in uh_u but then in terms of curvature *forms* one has:

$$F(\nabla^{uh_u}) = uF(\nabla^{h_u})$$

since the Levi-Civita connection of uh_u and h_u are the same. The reason for working with h_u is that we can make use of the results of [5], Ch. 10.

To begin with, let us notice that the family of vertical connections $\nabla^{VN}(u)$ resulting from the projections of the Levi-Civita connections ∇^{h_u} has a limit $\nabla^{VN}(0) := \lim\limits_{u\to 0}\nabla^{VN}(u)$ and this limit is the projection of the Levi-Civita connection of g^N onto VN. This follows from the Koszul formula (see also Prop. 10.2 in [5]).

Define, using the Ehresmann connection the following family of connections on $TN \to N$:

$$\nabla_u^{\oplus} := \nabla^{VN}(u) \oplus \pi^*\nabla^B \longrightarrow \nabla^{\oplus} := \nabla^{VN} \oplus \pi^*\nabla^B.$$

Remark 5.5. One has to be careful not to confuse ∇^{HN}, the result of projecting ∇^{h_u} onto HN, with $\pi^*\nabla^B$.

For $u \neq 0$, let $\tau_u : \Lambda^2 T^*N \to \mathrm{End}^-(TN)$ be the bundle morphism:

$$\tau_u(\omega_1\wedge\omega_2)(\xi) = \omega_2(\xi)\omega_1^{\sharp_u} - \omega_1(\xi)\omega_2^{\sharp_u}.$$

The notation $\sharp_u$ represents the g_u-metric dual. Notice that τ_u is the inverse of

$$(\tau_u)^{-1} : \mathrm{End}^-(TN) \to \Lambda^2 T^*N, \qquad (\tau_u)^{-1}(A)(\xi_1,\xi_2) = g_u(\xi_1, A\xi_2).$$

We can write (see Prop. 10.6 in [5]):

$$\nabla^{h_u} - \nabla_u^{\oplus} = \tau_u(\omega_u)$$

for $u \neq 0$, where $\omega_u : TN \to \Lambda^2 T^*N$ is defined by

$$\omega_u(X)(Y,Z) = \hat{S}_u(X,Y,Z) - \hat{S}_u(X,Z,Y) - \hat{\Omega}_u(X,Z,Y) + \hat{\Omega}_u(X,Y,Z) - \hat{\Omega}_u(Y,Z,X).$$

We recall the definitions of $\hat{S}_u$ and $\hat{\Omega}_u$ (both differ by a sign compared with Section 10.1 in [5]):

$$\hat{\Omega}_u \in \Gamma(HN^*\otimes HN^*\otimes VN^*), \qquad \hat{\Omega}_u(X,Y,Z) = \frac{1}{2}g_u^V(Z,[X,Y]),$$

$$\hat{S}_u \in \Gamma(VN^*\otimes VN^*\otimes HN^*), \qquad \hat{S}_u(X,Y,Z) = g_u^V(Y,[Z,X]^v - (\nabla^{VN}(u))_Z X).$$

where supscript ^{v} indicates projection onto the vertical component. Notice that both $\hat{\Omega}_u$ and $\hat{S}_u$ have well-defined limits when $u \to 0$. We conclude that ω_u has a well-defined limit ω_0 when $u \to 0$.

We look at the curvature tensors now. We get:

$$F(\nabla^{h_u}) = F(\nabla^{\oplus}_u) + [\nabla^{\oplus}_u, \tau_u(\omega_u)] + \tau_u(\omega_u) \wedge \tau_u(\omega_u). \tag{5.3}$$

Notice that for a fixed u, $\nabla^{\oplus}_u$ is h_u-metric compatible due to the fact that $\nabla^{VN}(u)$ is $g^V(u)$-metric compatible and $\pi^*\nabla^B$ is π^*g^B- metric compatible. As a consequence we have the following.

Lemma 5.6. *The morphism $\tau_u : \Lambda^2 T^*N \to \mathrm{End}^-(TN)$ is parallel with respect to the connection $\nabla^{\oplus}_u$ for every u.*

Proof. One proves directly that $\Theta^u := (\tau_u)^{-1}$ is parallel. □

Therefore

$$[\nabla^{\oplus}_u, \tau_u(\omega_u)] = \tau_u(\nabla^{\oplus}_u \omega_u). \tag{5.4}$$

where on the right $\nabla^{\oplus}_u$ is the extension on tensors of $\nabla^{\oplus}_u$. It preserves the type of a double form, i.e., it takes purely horizontal to purely horizontal etc.

Due to the fact that $\nabla^{HN} \neq \pi^*\nabla^B$, ω_u is not a mixed form, which means that $\tau_u(\omega_u)$ has a certain diagonal component. In fact we can write:

$$\omega_u = \tilde{\omega}_u + \omega^h_u \tag{5.5}$$

where $\tilde{\omega}_u$ is made exclusively of mixed terms while ω^h_u is a purely horizontal term with:

$$\tilde{\omega}_u := (\tau_u)^{-1}(\nabla^{h_u} - \nabla^{VN}(u) \oplus \nabla^{HN}(u))$$

and

$$\omega^h_u := (\tau_u)^{-1}(\nabla^{VN}(u) \oplus \nabla^{HN}(u) - \nabla^{\oplus}_u).$$

We used $\nabla^{HN}(u)$ for the horizontal orthogonal projection of ∇^{g_u} which does not coincide with $\pi^*\nabla^B$. Instead, we have the following.

Lemma 5.7. *Let $\pi : P \to B$ be a Riemannian submersion and let ∇^{HP} be the orthogonal projection of the Levi-Civita connection onto $HP \simeq \pi^*TB$. Let $\Omega : HP \times HP \to VP$*

$$\Omega(X, Y) = P^{VP}[X, Y]$$

be the curvature of the Ehresmann connection, a bundle morphism and $\widetilde{\Omega} : VP \times HP \to HP$ be the unique bundle morphism that satisfies

$$\langle \widetilde{\Omega}(X, Y), Z\rangle = \langle X, \Omega(Y, Z)\rangle, \qquad \forall Z \in \Gamma(HP).$$

Then, for all $X \in \Gamma(TP)$, $Y \in \Gamma(HP)$

$$\nabla^{HP}_X Y - (\pi^*\nabla^B)_X Y = \frac{1}{2}\Omega(P^{HP}(X), Y) - \frac{1}{2}\widetilde{\Omega}(P^{VP}(X), Y).$$

In particular

$$\langle \nabla^{HP}_X Y, Z\rangle - \langle \pi^*\nabla^B_X Y, Z\rangle = -\frac{1}{2}\langle P^{VP}(X), [Y, Z]\rangle, \qquad \forall Y, Z \in \Gamma(HP).$$

Proof. It is well-known (see [23], pag 82) that if X and Y are horizontal lifts of vector fields $\overline{X}, \overline{Y}$ on B then

$$\nabla^P_X Y = \pi \circ \nabla^B_{\overline{X}} \overline{Y} + \frac{1}{2}\Omega(X, Y).$$

In other words for this kind of vector fields one has:

$$\nabla^P_X Y = (\pi^*\nabla^B)_X Y + \frac{1}{2}\Omega(X, Y). \tag{5.6}$$

It is easy to extend equation (5.6) to vector fields $X = fX_1$ and $Y = gY_1$ where X_1 and Y_1 are horizontal lifts and $f, g \in C^\infty(P)$. This means that (5.6) holds for all $X, Y \in \Gamma(HN)$.

On the other hand, for $X \in \Gamma(VN)$ and Y, Z horizontal lifts, one has

$$2\langle \nabla^P_X Y, Z\rangle = \langle [X,Y],Z\rangle - \langle [Y,Z],X\rangle + \langle [Z,X],Y\rangle + X\langle Y,Z\rangle = -\langle [Y,Z],X\rangle$$

the reason being that $[X,Y] = 0 = [Z,X]$ (see Lemma 10.7 in [5]). Since, in this case $\pi^*\nabla^B_X Y = 0$ we get

$$\langle \nabla^P_X Y, Z\rangle - \langle \pi^*\nabla_X Y, Z\rangle = -\frac{1}{2}\langle [Y,Z],X\rangle \tag{5.7}$$

and the relation holds also for $Y = gY_1$ and $Z = hZ_1$ with Y_1 and Z_1 horizontal lifts and $g, h \in C^\infty(P)$. This means that (5.7) holds for all $X \in \Gamma(VN)$, $Y, Z \in \Gamma(HN)$. $\square$

According to Lemma 5.7 for $X \in \Gamma(TN)$ and $Y, Z \in \Gamma(HN) = \Gamma(\pi^*TB)$ we have:

$$\big\langle \big(\nabla^{VN}(u) \oplus \nabla^{HN}(u) - \nabla^{\oplus}_u\big)_X Y, Z\big\rangle = \omega^h_u(X)(Y,Z) = -1/2h^V_u(P^{VN}(X),[Y,Z])$$

and thus ω^h_u has a well-defined limit when $u \to 0$. Since ω_u has a limit we deduce from (5.5) that $\tilde{\omega}_u$ has a limit when $u \to 0$. We conclude that

$$\nabla^{\oplus}_u \omega_u = \nabla^{\oplus}_u \tilde{\omega}_u + \nabla^{\oplus}_u \omega^h_u$$

is a decomposition into a purely mixed term and a purely horizontal term since $\nabla^{\oplus}_u$ preserves the type of the form. Both sides have a well-defined limit when $u \to 0$.

In order to say something about $(\tau_u)^{-1}(\tau_u(\omega_u) \wedge \tau_u(\omega_u))$ we need to take a closer look at τ_u. Since for every $\eta \in \Omega^1(TN)$ we have

$$\eta^{\sharp_u} = (\eta^v)^{\sharp^v_u} + u(\eta^h)^{\sharp^h}$$

where the decomposition $\eta = \eta^v + \eta^h$ is independent of u and $\sharp^v_u$ is the g^V_u-metric dual while $\sharp^h$ is the π^*g^B-metric dual we get:

$$\tau_u = \tau^u_0 + u\tau'_0,$$

where

$$\tau^u_0 : \Lambda^2T^*N \to \mathrm{Hom}(TN, VN), \qquad \tau^u_0(\omega_1 \wedge \omega_2)(\xi) = \omega_2(\xi)(\omega^v_1)^{\sharp^v_u} - \omega_1(\xi)(\omega^v_2)^{\sharp^v_u}$$

$$\tau'_0 : \Lambda^2T^*N \to \mathrm{Hom}(TN, HN), \qquad \tau'_0(\omega_1 \wedge \omega_2)(\xi) = \omega_2(\xi)(\omega^h_1)^{\sharp^h} - \omega_1(\xi)(\omega^h_2)^{\sharp^h}.$$

Remark 5.8. Notice that if ω_1 or ω_2 is horizontal, then $\tau^u_0(\omega_1 \wedge \omega_2)\xi = 0$ for ξ vertical.

If ω_1 or ω_2 is vertical then $\tau'_0(\omega_1 \wedge \omega_2)\xi = 0$ for ξ horizontal.

If ξ is vertical but ω_1 and ω_2 are both horizontal then $\tau'_0(\omega_1 \wedge \omega_2)\xi = 0$.

Clearly $\tau_0(u)$ has a well-defined limit when $u \to 0$.

Let now $\gamma_u : \Lambda^2TN \to \Lambda^2T^*N$:

$$\gamma_u := (\tau_u)^{-1}(\tau_u(\omega_u) \wedge \tau_u(\omega_u)).$$

More explicitly

$$\begin{aligned}\gamma_u(a_1,a_2)(\xi_1,\xi_2) &= h_u\big(\xi_1, \tau_u(\omega_u(a_1))\tau_u(\omega_u(a_2)) - \tau_u(\omega_u(a_2))\tau_u(\omega_u(a_1))\xi_2\big) = \\ &= h_u\big(\tau_u(\omega_u(a_2))\xi_1, \tau_u(\omega_u(a_1))\xi_2\big) - h_u\big(\tau_u(\omega_u(a_1))\xi_1, \tau_u(\omega_u(a_2))\xi_2\big) = \\ &= g^V_u\Big(\tau^u_0(\omega_u(a_2))\xi_1, \tau^u_0(\omega_u(a_1))\xi_2\Big) - g^V_u\Big(\tau^u_0(\omega_u(a_1))\xi_1, \tau^u_0(\omega_u(a_2))\xi_2\Big) + \\ &\quad + u\Big[\pi^*g^B\big(\tau'_0(\omega_u(a_2))\xi_1, \tau'_0(\omega_u(a_1))\xi_2)\big) - \pi^*g^B\big(\tau'_0(\omega_u(a_1))\xi_1, \tau'_0(\omega_u(a_2))\xi_2)\big)\Big].\end{aligned}$$

The last equality is a consequence of the fact that $\tau_0(u)$ takes values in VN and τ'_0 takes values in HN.

We denote:

$$(\omega \wedge \omega)^u_0 : \Lambda^2TN \to \Lambda^2T^*N,$$

$$(\omega \wedge \omega)^u_0 := g^V_u\Big(\tau^u_0(\omega_u(a_2))\xi_1, \tau^u_0(\omega_u(a_1))\xi_2\Big) - g^V_u\Big(\tau^u_0(\omega_u(a_1))\xi_1, \tau^u_0(\omega_u(a_2))\xi_2\Big).$$

Remark 5.9. Notice that by Remark 5.8, τ_0^u will take mixed forms and purely horizontal forms into endomorphisms which vanish on vertical vectors. It is not hard to see (as in (5.5) below) that ω_u is a sum of mixed terms and purely horizontal terms. It follows that $\tau_0^u(\omega_u(a_2))\xi$ is zero for ξ vertical. We conclude that $(\omega\wedge\omega)_0^u$ is a purely horizontal form.

We denote

$$(\omega\wedge\omega)_0'(u):\Lambda^2TN\to\Lambda^2T^*N,$$
$$(\omega\wedge\omega)_0'(u):=\pi^*g^B\big(\tau_0'(\omega_u(a_2))\xi_1,\tau_0'(\omega_u(a_1))\xi_2\big)-\pi^*g^B\big(\tau_0'(\omega_u(a_1))\xi_1,\tau_0'(\omega_u(a_2))\xi_2\big)$$

Therefore

$$\gamma_u=(\omega\wedge\omega)_0^u+u(\omega\wedge\omega)_0'(u). \tag{5.8}$$

We will use the same notation $F(\nabla^{h_u})$ for the curvature forms $(\tau_u)^{-1}(F(\nabla^{h_u}))$ and $F(\nabla_u^{\oplus})$ for $(\tau_u)^{-1}(F(\nabla_u^{\oplus}))$. Then from (5.3), (5.4) and (5.8) we get the following equality of $(2,2)$ double forms:

$$F(\nabla^{g_u})=F(\nabla_u^{\oplus})+\nabla_u^{\oplus}\omega_u+(\omega\wedge\omega)_0(u)+u(\omega\wedge\omega)_0'(u).$$

The matrix decomposition $F(\nabla_u^{\oplus})=F(\nabla^{VN}(u))\oplus F(\pi^*\nabla^B)$ translates into the equality of $(2,2)$ double forms for the metric h_u:

$$F(\nabla_u^{\oplus})=F(\nabla^{VN}(u))+u^{-1}\pi^*F(\nabla^B).$$

We finally look at the decomposition for $(\omega\wedge\omega)_0'(u)$. Use (5.5) to get

$$(\omega\wedge\omega)_0'(u)=A_u^1+A_u^2+A_u^3+A_u^4,$$

where

$$\begin{aligned}
A_u^1(a_1,a_2)(\xi_1,\xi_2)&=\pi^*g^B(\tau_0'(\tilde\omega_u(a_2))\xi_1,\tau_0'(\tilde\omega_u(a_1))\xi_2)-\pi^*g^B(\tau_0'(\tilde\omega_u(a_1))\xi_1,\tau_0'(\tilde\omega_u(a_2))\xi_2)\\
A_u^4(a_1,a_2)(\xi_1,\xi_2)&=\pi^*g^B(\tau_0'(\omega_u^h(a_2))\xi_1,\tau_0'(\omega_u^h(a_1))\xi_2)-\pi^*g^B(\tau_0'(\omega_u^h(a_1))\xi_1,\tau_0'(\omega_u^h(a_2))\xi_2)\\
A_u^3(a_1,a_2)(\xi_1,\xi_2)&=\pi^*g^B(\tau_0'(\omega_u^h(a_2))\xi_1,\tau_0'(\tilde\omega_u(a_1))\xi_2)-\pi^*g^B(\tau_0'(\omega_u^h(a_1))\xi_1,\tau_0'(\tilde\omega_u(a_2))\xi_2)\\
A_u^2(a_1,a_2)(\xi_1,\xi_2)&=\pi^*g^B(\tau_0'(\tilde\omega_u(a_2))\xi_1,\tau_0'(\omega_u^h(a_1))\xi_2)-\pi^*g^B(\tau_0'(\tilde\omega_u(a_1))\xi_1,\tau_0'(\omega_u^h(a_2))\xi_2).
\end{aligned}$$

Now A_u^1 is purely vertical, A_u^4 is purely horizontal, and moreover one can check that A_u^2 and A_u^3 are mixed. We have thus proved the following

Proposition 5.10. *The following equality of* $(2,2)$ *double forms holds*

$$F(\nabla^{h_u})=\Big[u^{-1}\pi^*F(\nabla^B)+\nabla_u^{\oplus}\omega_u^h+(\omega\wedge\omega)_0^u+uA_u^4\Big]+\big[\nabla_u^{\oplus}\tilde\omega_u+uA_u^2+uA_u^3\big]+\big[F(\nabla^{VN}(u))+uA_u^1\big]$$

where the sums in square brackets represent the purely horizontal, mixed or purely vertical components.

All terms dependent on u *have a well-defined limit when* $u\to 0$.

From $F(\nabla^{uh_u})=uF(\nabla^{h_u})$ *one gets the corresponding decomposition for* $F(\nabla^{uh_u})$.

5.2. **Horizontal variations of the model metric.** We close this section by discussing what happens when the model metric has the following structure:

$$g_e=dr^2\oplus r^2g^V(r)\oplus\pi^*g^B(r)$$

with $g^B(r)$ a smooth family of metrics on $(-\epsilon,0]$. Different types of perturbations will be considered in Section 7.

By reasoning exactly as in the proof of Theorem 1.2 one can compute the limits of transgression forms. In order to state the result we need some notation.

Let $(g_r)_{r\in(-\epsilon,\epsilon)}$ be a smooth family of metrics on a smooth manifold B of dimension b. Let $g:=g_0$ and $\dot g:=\frac{\partial g}{\partial r}(0)$ and denote:

$$Q_{i,b}(g_r):=\frac{1}{i!(b-2i)!}\mathcal{B}_g\left(R^i\wedge\dot g^{b-2i}\right).$$

Theorem 5.11.

$$(2\pi)^k\chi(M) = \int_M \mathrm{Pf}^g - \sum_{(i,j)\in A_{k,b}} (-1)^{2k-b}\tilde{c}(k-j-1)\int_B \left(Q_{i,b}(g^B(r))\int_{N/B} P_{j,f}\left(g^V\right)\right)$$

where

$$A_{k,b} := \{(i,j) \mid 0 \le i \le j \le k-1,\ i \le b/2\}.$$

Proof. One writes $\mathrm{II} = -(rZ+T)$ where $T = \dot{g}^B$ and notices first that $ZT = TZ$. Then one ends up with a sum for fixed $0 \le j \le k-1$

$$\sum_i \sum_l (-1)^{f+1} r^{b-(2i+l)} \binom{j}{i}\binom{2k-2j-1}{l} \mathcal{B}_{g^N}\left(X^i(0)T^l(0)Y^{j-i}(0)Z^{2k-2j-1-l}(0)\right)$$

where $X(0)$ and $T(0)$ are purely horizontal. Only when $2i+l=b$ one gets something non-trivial. Multiply by $c(j,k)$ and sum to get the desired formula. ☐

Corollary 5.12. *If $\dot{g}^B(0) \equiv 0$ one recovers the formula of Theorem 1.2.*

Anticipating Section 7 we see that Theorem 5.11 is an example of a Gauss-Bonnet formula for first order perturbations of the model metric

$$dr^2 \oplus r^2 g^V(r) \oplus \pi^* g^B(0)$$

in the sense of Definition 7.6.

6. Manifolds with fibered boundary

The computations of the previous section allow us to address the Gauss-Bonnet problem for another class of metrics. Assume again that N fibers over B and that we fix an Ehresmann connection. Then consider the model metric on $(1,\infty)\times N$ of type:

$$g_e^\infty := dr^2 \oplus g^V \oplus r^2 \pi^* g^B.$$

We will consider Riemannian manifolds (M,g) and call them manifolds with fibered boundary for which there exists a diffeomorphism $\varphi : M \setminus K \to (1,\infty)\times N$ outside a compact set K such that

$$g = \varphi^* g_e^\infty.$$

Proposition 6.1. *A manifold with fibered boundary is complete.*

Proof. Outside a relatively compact set, M is isometric to $[r,\infty)\times N$ endowed with the metric g_e^∞ for some $r \in \mathbb{R}$. The projection onto $[r,\infty)$ is proper because N is compact. Moreover, this projection clearly decreases lengths of vectors, hence of curves, hence it decreases distances (it is Lipschitz of constant 1). This is enough to imply that $[r,\infty)\times N$ is a complete metric space, hence M is also complete. ☐

Proof of Theorem 1.3. The computations are similar to Theorem 1.2 and based also on Proposition 5.10 where we set $u^{-1} = r^2$. Let $g_r := g^V \oplus r^2\pi^* g^B = h_u$ be the metric of the slice. Write the decomposition in purely horizontal, mixed and purely vertical terms as:

$$F(\nabla^{g_r}) = (r^2 A_2 + A_0 + r^{-2}A_{-2}) + (C_0 + r^{-2}C_{-2}) + (D_0 + r^{-2}D_{-2})$$

where $A_2 = \pi^* F(\nabla^B)$, $D_0 = F(\nabla^{VN})$.

Then

$$(\mathrm{II}^r)^{2k-1-2j} = -r^{2k-1-2j}(\pi^* g^B)^{2k-1-2j}$$

and $\mathcal{B}_{g_r}(\cdot) = r^{-b}\mathcal{B}_{g^N}(\cdot)$ where $g^N = g^V \oplus \pi^* g^B$. Hence

$$\mathcal{B}_{g_r}\left(F(\nabla^{g_r})^j \wedge (\mathrm{II}^r)^{2k-1-2j}\right) = -r^{f-2j}\mathcal{B}_{g^N}\left(F(\nabla^{g_r})^j \wedge (\pi^* g^B)^{2k-1-2j}\right).$$

We look at the term $(r^2A_2)^l$ for some $l \leq j$ in the expansion of $F^{\nabla^{g_r}}$. Now the horizontal component of the product $F(\nabla^{g_r})^j \wedge (\pi^* g^B)^{2k-1-2j}$ cannot have degree bigger than b in order to be non-zero. Hence

$$2l + 2k - 1 - 2j \leq b \;\Leftrightarrow\; 2l + f - 2j \leq 0.$$

All the other terms in the expansion of $F(\nabla^{g_r})$ contribute with non-positive power of r. Hence in the expansion of $r^{f-2j}\mathcal{B}_{g^N}\left(F(\nabla^{g_r})^j \wedge (\pi^* g^B)^{2k-1-2j}\right)$ one ends up only with non-positive powers of r.

If b is even the inequalities are strict so all terms will vanish when $r \to \infty$. If b is odd, collecting the terms that correspond to $2l = 2j - f$ (which incidentally forces $j \geq f/2$) we get (1.3). ☐

Corollary 6.2. *If the basis of the fibration $N \to B$ is the odd-dimensional sphere with the round metric then*

$$\chi(M) - \chi(F) = \frac{1}{(2\pi)^k}\int_M \mathrm{Pf}^g.$$

Proof. The direction of the normal ∂_r points towards the outside of round sphere. Hence the computations of Example 3.4 apply (see also Remark 4.8). This fits with the example of $M = \mathbb{R}^n$ and F reduced to a point. ☐

7. Edge manifods: perturbations of the model metrics

There is one familiar situation which is not covered by the models of Section 5, namely that of a submanifold B of a Riemannian manifold (M, g). The spherical normal bundle $N := S\nu B$ inherits a fiber bundle structure over B and an Ehresmann connection, induced by the Levi-Civita connection as follows. Let $\pi : TN \to N$ be the natural projection. The Levi-Civita connection induces a connection on νB and therefore one obtains a splitting $T\nu B = \pi^*\nu B \oplus \pi^* TB$ into vertical and horizontal components where $\pi : \nu B \to B$ is the natural projection. Now $S(\nu B) \subset \nu B$ is a hypersurface whose unit normal vector is vertical (i.e., it belongs to $\pi^*\nu B$) relative to the previous decomposition. It follows that $TS(\nu B)$ splits into the direct sum of $\tau^\perp \subset \pi^*\nu B$ (the orthogonal complement of the tautological section of $\pi^*\nu B \to S(\nu B)$) and $\pi^* TB$.

On both TB and the normal vector bundle $\nu B \to B$ there are metrics induced by g, hence $(-\epsilon, 0) \times N$ inherits an edge singularity metric. However, the original metric g in a neighborhood of B is not necessarily isometric to a model metric in the sense defined in Section 5 since the normal exponential map that gives rise to a tubular neighborhood for B is only an "infinitesimal" isometry at the 0 section.

We therefore have to consider perturbations of the model edge metrics of Section 5.

We will consider a differentiable edge manifold, meaning a compact manifold M with boundary N, such that $\pi : N \to B$ is a local trivial fibration. Moreover we assume the following data given:

(a) a boundary defining function $r : M \to (-\epsilon, 0]$;
(b) an Ehresmann connection on N, i.e., a splitting $TN = VN \oplus \pi^* TB$

We can use r in order to produce a collar neighborhood U of N diffeomorphic with $(-\epsilon, 0] \times N$ such that the obvious diagram commutes:

$$\begin{tikzcd}
U \arrow[rr, "R", "\sim"'] \arrow[dr, "r"] & & (-\epsilon, 0] \times N \arrow[dl, "p_1"] \\
& (-\epsilon, 0] &
\end{tikzcd}$$

The differential of R gives a diffeomorphism between $TM\big|_U$ and $\mathbb{R} \oplus \pi_2^* TN$ where $\pi_2 : (-\epsilon, 0] \times N \to N$ is the second projection.

For our purposes, the edge manifold M in the neighborhood U will be identified with $(-\epsilon, 0] \times N$ while the tangent bundle to M in a neighborhood U will be identified with $\mathbb{R} \oplus \pi_2^* TN$. The unit generator of $\mathbb{R}$ in this identification will be denoted ∂_r.

For the sake of notation we will therefore sometimes write U for $(-\epsilon, 0] \times N$.

Consider the vector bundles $F := VN$ and $F' := \pi^*TB \oplus \mathbb{R}$ over N. Notice that the Ehresmann connection induces a splitting

$$\mathbb{R} \oplus TN \simeq F \oplus F'.$$

We use the projection $\pi_2 : (-\epsilon, 0] \times N \to N$ to pull-back this bundle to U but rather than writing π_2^*F, π_2^*F' we keep the notation F, F'. We have thus in the neighborhood U a splitting

$$TM\big|_U \simeq F \oplus F' \tag{7.1}$$

The fundamental object of this section is the following bundle endomorphism defined in terms of the splitting (7.1).

$$\varphi : TM\big|_U \to TM\big|_U, \qquad F \oplus F' \ni (v, w) \xrightarrow{\varphi} (rv, w).$$

Clearly, φ is a bundle isomorphism only along $U^c := U \setminus N$, i.e., for $r \neq 0$.

The model edge degenerate metric is throughout this section:

$$h := dr^2 \oplus r^2 g^V \oplus \pi^* g^B.$$

Theorem 7.1. *The bilinear map*

$$h^\varphi : TM\big|_{U^c} \times TM\big|_{U^c} \to \mathbb{R}, \qquad h^\varphi(Y', Z') := h(\varphi^{-1}(Y'), \varphi^{-1}(Z'))$$

extends as a non-degenerate metric on U and the map φ becomes a bundle isometry for $r \neq 0$. Moreover, the Levi-Civita connection ∇^h of the model metric has the property that $\varphi \nabla \varphi^{-1}$ extends to a h^φ-metric connection.

Proof. To first statement is obvious:

$$h^\varphi = dr^2 \oplus g^V \oplus \pi^* g^B.$$

For the second part we need a detailed description of ∇^h.

We will compare the Levi-Civita connection of ∇^h with the following connection

$$\nabla' := d \oplus \left[\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) dr + \nabla^{VN}\right] \oplus \pi_2^* \pi^* \nabla^B \tag{7.2}$$

on the vector bundle $TM\big|_{U^c} = \mathbb{R} \oplus \pi_2^* VN \oplus \pi_2^* \pi^* TB$ where $\pi_2 : (-\epsilon, 0] \times N \to N$ is the projection.

In (7.2), the connection ∇^{VN} is the projection of the Levi-Civita connection of any slice $\{r\} \times N$ onto $\pi_2^* VN$. It does not depend on r and this can be seen by remembering that the projection of the Levi-Civita connection of a Riemannian submersion onto the vertical bundle does not depend on the choice of the horizontal metric (Prop. 10.2 in [5]) while the Levi-Civita connection of the slice $\{r\} \times N$ is the same for the metric $r^2 g^V \oplus \pi^* g^B$ as for the metric $g^V \oplus r^{-2} \pi^* g^B$.

We emphasize that the differential operator $\frac{\partial}{\partial r} + \frac{1}{r}$ acts on families of sections

$$(Y_r)_{r \in (-\epsilon, 0]} \in \Gamma(VN)$$

which can alternatively be seen as sections of $\pi_2^* VN$ where $\pi_2 : (-\epsilon, 0] \times N \to N$ is the projection, while ∇^{VN} is used to differentiate only in the TN directions.

It follows from the Koszul relation (see (7.4) and (7.5)) that the $\pi_2^* VN$ component of ∇' is actually the orthogonal projection of ∇^h onto $\pi_2^* VN$ and this implies that ∇' is h-compatible (as $\pi^* \nabla^B$ is clearly $\pi^* g^B$ compatible). As a consequence, $\varphi \nabla' \varphi^{-1}$ is h^φ compatible.

It is easy to check that $\varphi \nabla' \varphi^{-1}$ extends to $r = 0$ since ∇^{VN} commutes with multiplication by r^{-1} and

$$\frac{\partial Y_r}{\partial r} + \frac{Y_r}{r} = \frac{1}{r} \frac{\partial (r Y_r)}{\partial r}$$

Moreover $\varphi(d \oplus \pi^* \nabla^B) \varphi^{-1} = d \oplus \pi^* \nabla^B$ as φ acts as identity on F'.

In order for the 1-form $\eta := \nabla^h - \nabla'$ to have the property that $\varphi\eta(X)\varphi^{-1}$ extends smoothly for every choice of $X \in \Gamma(TM\big|_U)$ it is enough that in the decomposition

$$\eta(X) := \begin{pmatrix} A_1(X) & A_2(X) \\ A_3(X) & A_4(X) \end{pmatrix} : \begin{matrix} F \\ \oplus \\ F' \end{matrix} \to \begin{matrix} F \\ \oplus \\ F' \end{matrix} \tag{7.3}$$

the blocks $A_i(X)$, $i = 1, 4$ extend smoothly at $r = 0$, $rA_2(X)$ extends smoothly and $A_3(X) = rC_3(X)$ for some $C_3(X)$ smooth, all the way up to $r = 0$.

Clearly $A_1 \equiv 0$ since the orthogonal projections of ∇^h and ∇' on F coincide.

Then metric compatibility implies for $Y \in \Gamma(F')$ and $Z \in \Gamma(F)$

$$\begin{aligned} r^2\langle A_2(X)(Y), Z\rangle_{VN} = \langle A_2(X)(Y), Z\rangle_h = -\langle Y, A_3(X)(Z)\rangle_h = \\ = -\langle Y, A_3(X)(Z)\rangle_{F'} = -\langle A_3^T(X)(Y), Z\rangle_{VN} \end{aligned}$$

where the transpose A_3^T is computed in the metric h^φ which is independent of r. Hence

$$rA_2(X) = -\frac{A_3^T(X)}{r}.$$

We conclude that it is enough to prove that $\frac{A_3(X)}{r}$ extends.

To see that the rest holds we look again at the Koszul relation:

$$2\langle \nabla^h_X Y, Z\rangle_h = \langle [X,Y], Z\rangle_h - \langle [Y,Z], X\rangle_h + \langle [Z,X], Y\rangle_h + X\langle Y, Z\rangle_h + Y\langle Z, X\rangle_h - Z\langle X, Y\rangle_h$$

If $X = \partial_r$, $Y = Y_r \in \Gamma(\pi_2^* VN)$, $Z = Z_r \in \Gamma(\pi_2^* VN)$ then

$$2r^2 \langle \nabla_{\partial_r} Y, Z\rangle_{g^{VN}} = r^2 \left\langle \frac{\partial Y}{\partial r}, Z \right\rangle_{g^{VN}} - r^2 \left\langle \frac{\partial Z}{\partial r}, Y \right\rangle_{g^{VN}} + \frac{\partial}{\partial r}\left[r^2 \langle Y, Z\rangle_{g^{VN}} \right]$$

We end up with

$$2r^2 \langle \nabla_{\partial_r} Y, Z\rangle_{g^{VN}} = 2r^2 \left\langle \frac{\partial Y}{\partial r}, Z \right\rangle_{g^{VN}} + 2r \langle Y, Z\rangle_{g^{VN}}$$

Hence

$$\nabla_{\partial_r} Y = \frac{\partial Y}{\partial r} + \frac{Y}{r} \tag{7.4}$$

Taking $X = X_r \in \Gamma(\pi_2^* TN)$ with $Y = Y_r \in \Gamma(\pi_2^* VN)$, $Z = Z_r \in \Gamma(\pi_2^* VN)$ then clearly

$$\langle \nabla^h_X Y, Z\rangle = \langle \nabla^{VN}_X Y, Z\rangle \tag{7.5}$$

One verifies easily that the orthogonal projection of ∇^h onto $\mathbb{R}$, the tangent bundle of the foliation via integral curves of ∂_r is d.

Recall that $\pi^*\nabla^B$ is not the orthogonal projection of ∇^h onto $\pi^*TB \simeq HN$. Let ∇^{HN} be this projection. It follows from Lemma 5.7 for the Riemannian submersion $M\big|_U \to B$ that for $X \in \Gamma(\mathbb{R} \oplus \pi_2^* TN)$ and $Y,\ Z \in \Gamma(\pi_2^*\pi^* TB)$:

$$\langle \nabla^h_X Y, Z\rangle_h = \langle \nabla^{HN}_X Y, Z\rangle_h = \left\langle \pi^*\nabla^B_X Y, Z\right\rangle_h - \frac{1}{2}\left\langle P^{\mathbb{R}\oplus VN}(X), [Y,Z]\right\rangle_h$$

When $X = \partial_r$ since $[Y, Z] \in \Gamma(\pi_2^* TN)$ (one has a foliation via hypersurfaces $\{r\} \times N$) the last term is zero.

When $X \in \Gamma(\pi_2^* TN)$ then

$$\langle \nabla^h_X Y, Z\rangle_h = \left\langle \pi^*\nabla^B_X Y, Z\right\rangle_h - \frac{r^2}{2}\left\langle P^{VN}(X), [Y,Z]\right\rangle_{g^{VN}}$$

and the right hand side makes sense at $r = 0$. This describes the bottom block diagonal component of $A_4(X)$ in (7.3) relative the decomposition $F' = \mathbb{R} \oplus \pi^*TB$. The other diagonal block of A_4 is obviously 0. The off-diagonal terms of the skew-symmetric $A_4(X)$ are of type

$$\langle \nabla_X \partial_r, Y\rangle_h \text{ and its negative } \langle \nabla_X Y, \partial_r\rangle_h$$

where $X \in \Gamma(\mathbb{R} \oplus \pi_2^* TN)$, $Y \in \Gamma(\pi_2^* HN)$. For $X = \partial_r$ one gets obviously 0 and Lemma 3.1 gives for $X \in \Gamma(\pi_2^* TN)$:

$$\langle \nabla_X \partial_r, Y \rangle_h = \frac{1}{2}(L_{\partial_r} h)(X, Y) = r\langle X, Y\rangle_{VN} = 0$$

In other words, if $\widetilde{\Omega} : VN \times HN \to HN$ is the morphism induced by the curvature Ω of the Ehresmann connection of the Riemannian submersion $\pi : N \to B$ with the metric $g^V \oplus \pi^* g^B$ as in Lemma 5.7, then for $X \in \Gamma(\mathbb{R} \oplus \pi_2^* TN)$, $Y \in \Gamma(F')$ one has:

$$A_4(X)(Y) = -\frac{r^2}{2}\widetilde{\Omega}(P^{VN}(X), P^{HN}(Y)).$$

Finally, for $Y \in \Gamma(\pi_2^* VN)$, $Z \in \Gamma(F')$ and $X \in \Gamma(\mathbb{R} \oplus \pi_2^* TN)$ we compute

$$\langle A_3(X)(Y), Z\rangle_h = \langle \nabla^h_X Y, Z\rangle_h.$$

For $X = \partial_r$, $Z \in \pi_2^* HN$ one gets from the Koszul formula

$$2\left\langle \nabla_{\partial_r} Y, Z\right\rangle_h = \left\langle \partial_r Y, Z\right\rangle_h - \left\langle \partial_r Z, Y\right\rangle_h = 0. \tag{7.6}$$

The vanishing stays true also for $X = \partial_r$, $Z = \partial_r$.

For $X \in \Gamma(\pi_2^* TN)$, $Z = \partial_r$ we get:

$$\langle A_3(X)(Y), \partial_r\rangle = \langle \nabla^h_X Y, \partial_r\rangle_h = \mathrm{II_r}(X, Y) = -r\langle X, Y\rangle_{VN}. \tag{7.7}$$

For $X \in \Gamma(\pi_2^* VN)$, $Z \in \Gamma(\pi_2^* HN)$ we get the evolution with of the second fudamental form of the leaves of the fibration $\pi : N \to B$:

$$\langle A_3(X)(Y), Z\rangle_{HN} = -\langle Y, P^{VN}(\nabla^h_X Z)\rangle_h = r^2\langle Y, P^{VN}([Z, X]) - \nabla^{VN}_Z X\rangle_{VN} \tag{7.8}$$

For $X \in \Gamma(\pi_2^* HN)$, $Z \in \Gamma(\pi_2^* HN)$ we get the curvature of the Ehresmann connection:

$$\langle A_3(X)(Y), Z\rangle_{HN} = -\langle Y, P^{VN}(\nabla^h_X Z)\rangle_h = -r^2\langle Y, \Omega(X, Z)\rangle_{VN}. \tag{7.9}$$

It is now clear from (7.6), (7.7), (7.8) and (7.9) that $\frac{A_3(X)}{r}$ extends for any smooth vector fields $X, Y : (-\epsilon, 0] \times N \to TM\big|_U$. ☐

Corollary 7.2. *The extended connection $\varphi\nabla^h\varphi^{-1}$ has the property that when it is restricted to $TM\big|_{\partial M} = \mathbb{R} \oplus \pi_2^* VN \oplus \pi_2^*\pi^* TB$, i.e., to $r = 0$ it coincides with the connection*

$$\begin{pmatrix} d \\ \frac{\partial}{\partial r} dr + \nabla^{VN} \\ \pi^*\nabla^B \end{pmatrix} + \left(\begin{array}{c|c|c} 0 & \langle \bullet, \cdot\rangle_{VN} & 0 \\ -\langle \bullet, \cdot\rangle_{VN} & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \tag{7.10}$$

where the matrix represents a 1*-form (the* $\bullet$ *entry) with values in* $\mathrm{End}(\mathbb{R} \oplus \pi_2^* VN \oplus \pi_2^*\pi^* TB)$*.*

Proof. The only non-trivial term in the difference $\varphi(\nabla^h - \nabla')\varphi^{-1}$ comes from relation (7.7). ☐

Corollary 7.3. *The Pfaffian* $\mathrm{Pf}(\nabla^h)$ *is a smooth form on M.*

Proof. The map $\varphi : (TM\big|_{U^c}, h) \to (TM\big|_{U^c}, h^\varphi)$ is a bundle isometry. Hence on U^c, $\mathrm{Pf}(\nabla^h)$ is, up to a sign, equal to $\mathrm{Pf}(\varphi\nabla^h\varphi^{-1})$. ☐

We consider now a perturbation g of h, i.e., a bilinear and symmetric form on TM that is degenerate only along N in a sense made precise in Definition 7.6.

Clearly there exists an h-symmetric endomorphism $C \in \Gamma(\mathrm{End}(TM\big|_{U^c}))$ such that

$$g(X, Y) = h(CX, Y) = h(X, CY), \qquad \forall X, Y \in TM\big|_{U^c}.$$

The next Lemma that connects the two Levi-Civita connections is fundamental.

Lemma 7.4 (Christoffel formula)**.** *Let* ∇^h *and* ∇^g *be the corresponding Levi-Civita connections on* $TM\big|_{U^c}$*. Then the* 1*-form* $\omega : TM\big|_{U^c} \to \mathrm{End}(TM\big|_{U^c})$ *defined by*

$$\omega(X)(Y) = \nabla^g_X Y - \nabla^h_X Y,$$

satisfies:

$$h(C\omega(X)(Y), Z) = \frac{1}{2}\Big(h((\nabla^h_X C)Y, Z) + h((\nabla^h_Y C)X, Z) - h((\nabla^h_Z C)X, Y\Big).$$

Proof. Notice first that due to the symmetry of the Levi-Civita connections one has:

$$\omega(X)(Y) = \omega(Y)(X) \tag{7.11}$$

and therefore $C\omega(X)(Y) = C\omega(Y)(X)$. Then from

$$Xh(Y, CZ) = h(\nabla^h_X Y, CZ) + h(Y, \nabla^h_X (CZ)) \text{ and}$$
$$Xg(Y, Z) = g(\nabla^g_X Y, Z) + g(Y, \nabla^g_X Z)$$

which translates into

$$Xh(Y, CZ) = h(\nabla^g_X Y, CZ) + h(Y, C\nabla^g_X Z)$$

one gets by subtraction:

$$h(\nabla^h_X Y - \nabla^g_X Y, CZ) = h(Y, C\nabla^g_X Z - \nabla^h_X (CZ)).$$

Taking $\nabla^h_X (CZ) = C(\nabla^h_X Z) + (\nabla^h_X C)(Z)$ we get:

$$h(\omega(X)(Y), CZ) + h(Y, C\omega(X)(Z)) = h(Y, (\nabla^h_X C)(Z)).$$

or

$$\omega(X)^T C + C\omega(X) = \nabla^h_X C. \tag{7.12}$$

Notice that the system (7.11) and (7.12) has a unique solution for $C\omega(X)$ due to the well-known fact that a trilinear map which is symmetric in the first two variables and anti-symmetric in the last two variables is zero. Finding this solution is simple linear algebra. ☐

We know already that $\varphi\nabla^h\varphi^{-1}$ extends to $r = 0$. Let

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} F \\ F' \end{bmatrix} \to \begin{bmatrix} F \\ F' \end{bmatrix}$$

be the block decomposition of C then

$$C^\varphi := \varphi C \varphi^{-1} = \begin{bmatrix} C_1 & rC_2 \\ r^{-1}C_3 & C_4 \end{bmatrix}$$

is symmetric with respect to the h^φ-metric. In other words $C_3 = r^2 C_2^T$ where the transpose is computed with respect to h^φ. We have the following obvious remark.

Lemma 7.5. *If* $\varphi C\varphi^{-1}$ *extends smoothly to* $TM\big|_U$*, then* $g^\varphi(\cdot,\cdot) := g(\varphi^{-1}(\cdot), \varphi^{-1}(\cdot))$ *extends and*

$$g^\varphi(\cdot,\cdot) = h^\varphi(C^\varphi\cdot,\cdot).$$

The morphism $\varphi C\varphi^{-1}$ controls the degenerations we would like to consider. Notice that by Lemma 7.5 saying that

$$C = I + f(r)\varphi^{-1}D\varphi$$

where D is smooth at $r = 0$ and h^φ-symmetric and f smooth and vanishing at 0 is equivalent to saying that

$$g^\varphi(\cdot,\cdot) = h^\varphi(\cdot,\cdot) + f(r)\alpha(\cdot,\cdot)$$

for some $\alpha(\cdot,\cdot)$ smooth, bilinear, symmetric on $TM\big|_U$.

Definition 7.6. A perturbation of first (respectively second) order of h is a bilinear, positive, symmetric $g : TM\big|_{U^c} \times TM\big|_{U^c}$ such that the endomorphism C above satisfies:

$$C = I + r\varphi^{-1}D\varphi, \quad \text{resp.} \quad C = I + r^2\varphi^{-1}D\varphi,$$

where D is a smooth endomorphism of $TM\big|_U$, symmetric in the h^φ metric.

Equivalently for $p = 1$ (resp. $p = 2$)

$$g^\varphi(\cdot,\cdot) = h^\varphi(\cdot,\cdot) + r^p\alpha(\cdot,\cdot)$$

where α is bilinear, symmetric and smooth on $TM\big|_U$.

Lemma 7.7.

$$\varphi(\nabla^h C)\varphi^{-1} = (\varphi\nabla^h\varphi^{-1})(\varphi C\varphi^{-1})$$

Proof. It follows from the next equalities that hold for any X and Y:

$$\varphi(\nabla^h_X C)\varphi^{-1}(Y) = \varphi(\nabla^h_X(C\varphi^{-1}(Y)) - \varphi C(\nabla^h_X(\varphi^{-1}(Y)))$$

$$(\varphi\nabla^h_X\varphi^{-1})(\varphi C\varphi^{-1})(Y) = \varphi(\nabla^h(\varphi^{-1}\varphi C\varphi^{-1}(Y))) - \varphi C\varphi^{-1}(\varphi\nabla^h(\varphi^{-1}(Y))).$$

☐

Theorem 7.8. *Let g be a perturbation of a model edge metric h.*

(i) *For perturbations of first order, the connection $\varphi\nabla^g\varphi^{-1}$ extends at $r = 0$.*

(ii) *For pertubations of second order the connection the extension of $\varphi\nabla^g\varphi^{-1}$ coincides on $TM\big|_{\partial M}$ with $\varphi\nabla^h\varphi^{-1}$.*

Proof. Let $C^\varphi := \varphi C\varphi^{-1}$, $Y' := \varphi(Y)$, $Z' := \varphi(Z)$, $\nabla^\varphi := \varphi\nabla^h\varphi^{-1}$, $\omega(X)^\varphi := \varphi\omega(X)\varphi^{-1}$

$$h^\varphi(\cdot,\cdot) := h(\varphi^{-1}(\cdot),\varphi^{-1}(\cdot))$$

Then the Christoffel formula (Lemma 7.4) can be written using Lemma 7.7 as:

$$2h^\varphi\left(C^\varphi\omega(X)^\varphi(Y'), Z'\right) = h^\varphi\left((\nabla^\varphi_X C^\varphi)(Y'), Z'\right) + \tag{7.13}$$

$$+h^\varphi\left((\nabla^\varphi_{\varphi^{-1}(Y')}C^\varphi)(\varphi(X))), Z'\right) - h^\varphi\left((\nabla^\varphi_{\varphi^{-1}(Z')}C^\varphi)(\varphi(X))), Y'\right)$$

We deduce from this formula that in order to show that $\varphi\omega(X)\varphi^{-1}$ extends for perturbations of first order it is enough to show that

$$\nabla^\varphi_{\varphi^{-1}(Y')}C^\varphi = \nabla^\varphi_{\varphi^{-1}(Y')}(rD)$$

extends for all choices of Y', since the first term in the sum (r.h.s of (7.13))) extends anyway.

The only situation when the extension is not apriori clear is when $Y' \in \Gamma(\pi_2^*VN)$. Then $\varphi^{-1}(Y') = \frac{Y'}{r}$. But we can use now that $Y'(r) = 0$ and therefore

$$\nabla^\varphi_{\varphi^{-1}(Y')}(rD) = \nabla^\varphi_{Y'}(D),$$

and the later term extends.

Since $C^\varphi \to 0$ when $r \to 0$, in order to show that $\varphi\omega(X)\varphi^{-1}$ extends by 0 for perturbations of second order we need to check that

$$\lim_{r\to 0}\nabla^\varphi_X C^\varphi = \lim_{r\to 0}\nabla^\varphi_X(r^2D) = 0$$

$$\lim_{r\to 0}\nabla^\varphi_{\varphi^{-1}(Y')}C^\varphi = \lim_{r\to 0}\nabla^\varphi_{\varphi^{-1}(Y')}(r^2D) = 0$$

for all choices of X and Y'. If either $X = Y' = \partial_r$ then since $\varphi^{-1}(\partial_r) = \partial_r$ the two limits are identical and clearly equal to 0. When $Y' \in \Gamma(\pi_2^*VN)$ then the same idea as in the first order perturbations apply. ☐

Corollary 7.9. *For first- and second-order perturbations g of the model edge metrics h, the Pfaffian* Pf^g *is a smooth form on M.*

Proof. Analogous to Corollary 7.3. ☐

7.1. **The Riemannian metric in a neighborhood of a submanifold.**

The purpose of this Section is to prove that the degenerate metric on the oriented blow-up space of a submanifold inside a Riemannian manifold, is a normal, second order perturbation of a canonical model edge degenerate metric.

Let $B \subset M$ be a compact submanifold in a Riemannian manifold (M, g). Let $\nu B \subset TX_{|B}$ be the normal bundle, $\pi : S(\nu B) \to B$ the unit sphere bundle inside νB, and

$$\exp : S(\nu B) \times [0, \infty) \to M, \qquad (v_x, r) \mapsto \exp_x(rv_x)$$

the geodesic exponential map in normal directions to B. This map defines a diffeomorphism from $S(\nu B) \times (0, \epsilon)$ to the complement of B inside its ϵ-neighborhood. The function r becomes the distance function to B. In fact, replacing the ϵ-neighborhood of B with $S(\nu B) \times [0, \epsilon)$ amounts precisely to constructing the (real) blow-up of M along B.

The normal bundle νB inherits itself a metric which makes the canonical projection $\pi : \nu B \to B$ a Riemannian submersion. The Ehresmann connection here is just the normal connection on B induced from the Levi-Civita connection of M. One can use the blow-down map:

$$\exp : [0, \epsilon) \times S(\nu B) \to M$$

which is a diffeomorphism for $r \neq 0$ in order to endow $[0, \epsilon) \times S(\nu B)$ with a degenerate metric g_1. Clearly there exist a model edge degenerate metric h_1 on $[0, \epsilon) \times S(\nu B)$ of type:

$$dr^2 \oplus r^2 g^V \oplus \pi^* g^B$$

where g^V, the metric on $VS(\nu B) \subset \pi^* \nu B$ is induced by pulling back the metric $g\big|_{\nu B}$. The decomposition is relative to the Ehresmann connection mentioned earlier.

Theorem 7.10. *Let $B \subset M$ be a compact submanifold in a Riemannian manifold (M, g).*

Then the degenerate metric g_1 on $[0, \epsilon) \times S(\nu B)$ is a second order perturbation of the model degenerate metric h_1.

Proof. Due to Gauss Lemma we have that $R := \partial_r$ is a geodesic field, orthogonal to the slices $\{r\} \times S(\nu B)$ and therefore $g_1 = dr^2 \oplus g_1(r)$. We need only look at $g_1(r)$ on $T(S\nu B)$. The metric $g_1(r)$ is obtained via the map:

$$\exp^r : S\nu B \to M, \qquad (p, v) \to \exp_p(rv), \qquad g_1(r)(\cdot, \cdot) := g(d\exp^r(\cdot), d\exp^r(\cdot)).$$

We use curves $W : (-\epsilon, \epsilon) \to S(\nu B)$ with $\gamma(s) := \pi(W(s))$ where $\pi : S(\nu B) \to B$ is the projection in order to represent tangent vectors of $S(\nu B)$. Let then

$$f(r) := g_1(r)(W_1'(0), W_2'(0)) = g\left(\partial_s \exp^r(W_1(s))_{|s=0}, \partial_s \exp^r(W_2(s))_{|s=0}\right).$$

Notice that

$$J_i(r) := \partial_s \exp^r(W_i(s))_{|s=0}$$

are Jacobi vector fields, along the geodesics $r \to \exp_{\gamma_i(0)}(rW_i(0))$. We will assume that $W_1(0) = W_2(0) = (b, W) \in S(\nu_b B)$.

In order to make the computations more transparent it is useful to separate two classes of vector fields W along γ.

(a) the vertical ones, i.e., those for which $\gamma(s) \equiv b \in B$ is constant and therefore $J(0) = 0$ and $J'(0) = W'(0) \in T_{W(0)} S(\nu_b B)$ is a vertical vector in $T_{W(0)} S\nu B$.

(b) the horizontal ones, i.e., those for which $\nabla_{\gamma'} W \equiv 0$; these satisfy $J(0) = \gamma'(0)$ and $J'(0) = 0$; notice that the condition $\nabla_{\gamma'} W = 0$ implies that $W'(0)$ is a horizontal vector in $T_{W(0)} S\nu B$ such that $d\pi(W'(0)) = \gamma'(0)$.

By what was just said one has:

(a) when $W_1'(0)$, $W_2'(0)$ are both horizontal:

$$\begin{aligned} f(0) &= g(J_1(0), J_2(0)) = g(\gamma_1'(0), \gamma_2'(0)) = g(W_1'(0), W_2'(0)),\\ f'(0) &= \partial_r g(J_1(r), J_2(r))_{|r=0} = g(J_1'(0), J_2(0)) + g(J_1(0), J_2'(0)) = 0,\\ f''(0) &= \partial_r^2 g(J_1(r), J_2(r))_{|r=0} = \big[g(J_1''(r), J_2(r)) + 2g(J_1'(r), J_2'(r)) + g(J_1(r), J_2''(r))\big]_{|r=0}\\ &= \left[g\left(R^g(\partial_r, J_1(r))\partial_r, J_2(r)\right) + g\left(J_1(r), R^g(\partial_r, J_2(r))\partial_r\right)\right]_{|r=0} \end{aligned} \tag{7.14}$$

where we used that J_1 and J_2 are Jacobi.

(b) when $W_1'(0)$ is horizontal and $W_2'(0)$ vertical:

$$\begin{aligned} f(0) &= g(J_1(0), J_2(0)) = 0,\\ f'(0) &= g(J_1'(0), J_2(0)) + g(J_1(0), J_2'(0)) = 0,\\ f''(0) &= 0. \end{aligned}$$

The last equality holds because in (7.14), $J_2(0) = 0$.

(c) when $W_1'(0)$ and $W_2'(0)$ are both vertical:

$$\begin{aligned} f(0) &= 0 = f'(0),\\ f''(0) &= 2g(J_1'(0), J_2'(0)) = 2g(W_1'(0), W_2'(0)),\\ f'''(0) &= \partial_r\left[g\left(R^g(\partial_r, J_1(r))\partial_r, J_2(r)\right) + g\left(J_1(r), R^g(\partial_r, J_2(r))\partial_r\right)\right]\Big|_{r=0} = 0 \end{aligned}$$

again because $J_2(0) = 0$. Summarizing:

- for W_1, W_2 both horizontal, $g_1(r)(W_1, W_2) = g(W_1, W_2) + O(r^2)$;
- for W_1 horizontal and W_2 vertical, $g_1(r)(W_1, W_2) = O(r^3)$;
- for W_1 and W_2 both vertical, $g_1(r)(W_1, W_2) = r^2 g(W_1, W_2) + O(r^4)$.

Recall now that $g_1^\varphi(r)(W_1, W_2) = g_1(r)(P^H(W_1) + r^{-1}P^V(W_1), P^H(W_2) + r^{-1}P^V(W_2))$. We get that

$$g_1^\varphi(r)(W_1, W_2) = g(W_1, W_2) + O(r^2) = h_1^\varphi(W_1, W_2) + O(r^2)$$

and this corresponds to Definition 7.6. ☐

7.2. **Gauss-Bonnet for perturbations of model metrics.** We will look at perturbation of second order (Definition 7.6) of canonical model edge degenerate metrics. We assume again that M is an edge manifold.

A canonical model edge degenerate metric h is uniquely determined by the following data

(a) a collar neighborhood $U \supset \partial M$ with a diffeomorphism $R : U \to (-\epsilon, 0] \times N$ that makes the obvious diagram commutative;
(b) an Ehresmann connection on $\pi : \partial M = N \to B$;
(c) a metric g^V on $\operatorname{Ker} d\pi$;
(d) a metric g^B on B.

We would like to prove the following

Theorem 7.11. *Let g be a second order perturbation of a canonical model edge degenerate metric h. Then*

$$\lim_{r\to 0}\int_{\{r\}\times N} \mathrm{TPf}^g = \lim_{r\to 0}\int_{\{r\}\times N} \mathrm{TPf}^h. \tag{7.15}$$

Consequently, Gauss-Bonnet Theorem 1.2 holds verbatim where the odd Pfaffian form is associated to the degenerate metric h.

Proof. We use the notations of Section 7. One consequence of the definition of perturbation is that the bilinear form

$$g^\varphi(\cdot,\cdot) = g(\varphi^{-1}(\cdot), \varphi^{-1}(\cdot))$$

is a well-defined smooth metric on TM. Moreover if ∇^g is the Levi-Civita connection of g away from $r=0$, then $\varphi\nabla^g\varphi^{-1}$ is a g^φ-metric connection. As proved in Theorem 7.8 this connection is defined everywhere.

It is easy to check that if ∇^1 and ∇^2 are two g-metric compatible connections and is $\varphi : E \to E$ a bundle isometry where on the right one uses g^φ then

$$\mathrm{TPf}(\nabla^1,\nabla^2)=\mathrm{TPf}(\varphi\nabla^1\varphi^{-1},\varphi\nabla^2\varphi^{-1}).$$

This is the case for $E = TM\big|_{\{r\}\times N}$ with $r \neq 0$ and $\nabla^1 = \nabla^g$ and $\nabla^2 = d \oplus P\nabla^g P$ constructed as in Example 2.1. The fact that $\varphi\nabla^g\varphi^{-1}$ exists for all values of r implies immediately that the left hand side limit in (7.15) exists.

Moreover the limit is entirely determined by $\varphi\nabla^g\varphi^{-1}\big|_{TM|_{\partial M}}$ and the orthogonal decomposition

$$TM\big|_{\partial M} = \mathbb{R}\partial_r \oplus TN.$$

Due to the fact that $g^\varphi\big|_{TM|_{\partial M}} = h^\varphi\big|_{TM|_{\partial M}}$, the vector ∂_r has norm 1 also for g^φ at $r=0$.

A similar reasoning applies to the metrics h and h^φ.

Let us summarize. If $s_g(r)$ and $s_h(r)$ are the unit exterior normals to the slices $\{r\}\times N$ with respect to the metrics g^φ and h^φ, then

$$\mathrm{TPf}^g\big|_{\{r\}\times N} = \mathrm{TPf}(\varphi\nabla^g\varphi^{-1}, s_g(r)), \qquad \mathrm{TPf}^h\big|_{\{r\}\times N} = \mathrm{TPf}(\varphi\nabla^h\varphi^{-1}, s_h(r)).$$

We use now a similar trick as in Subsection 2.2, namely consider on $[0,1]\times U$, where U is the collar, the metric

$$ds^2 + (1-s)h^\varphi + sg^\varphi.$$

Parallel transport induces a bundle isometry

$$\tau_1^{-1} : (TM\big|_U, g^\varphi) \to (TM\big|_U, h^\varphi).$$

While parallel transport τ_1^{-1} need not take $s_g(r)$ to $s_h(r)$ since at $r=0$ $s_g(0)=s_h(0)$ and $\tau_1\big|_{TM|_{\partial M}} = \mathrm{id}$ it is clear that for r small one can find a smooth homotopy between $\tau_1^{-1}\circ s_g(r)$ and $s_h(r)$ within $(S(TM), h^\varphi)$. Then we can apply Proposition 2.8 to conclude that

$$\begin{aligned}
&\mathrm{TPf}(\varphi\nabla^g\varphi^{-1}, s_g(r)) - \mathrm{TPf}(\varphi\nabla^h\varphi^{-1}, s_h(r)) =\\
&= \mathrm{TPf}(\tau_1^{-1}\varphi\nabla^g\varphi^{-1}\tau_1, \tau_1^{-1}(s_g(r))) - \mathrm{TPf}(\varphi\nabla^h\varphi^{-1}, s_h(r)) =\\
&= -\mathrm{TPf}(\tau_1^{-1}\varphi\nabla^g\varphi^{-1}\tau_1, \varphi\nabla^h\varphi^{-1})\big|_{\{r\}\times N} + d\gamma.
\end{aligned}$$

Clearly $\int_{\{r\}\times N} d\gamma = 0$, while

$$\mathrm{TPf}(\tau_1^{-1}\varphi\nabla^g\varphi^{-1}\tau_1, \varphi\nabla^h\varphi^{-1})\big|_{\{0\}\times N} = 0$$

because τ_1 is the identity and for second order perturbations $\varphi\nabla^g\varphi^{-1}$ coincides with $\varphi\nabla^h\varphi^{-1}$ at $r=0$. □

7.3. **First order perturbations.** We start with the observation that the computations made in the proof of Theorem 7.11 justify the following:

Theorem 7.12. *Let g be a first order perturbation of a model edge metric $h = dr^2 \oplus r^2h^V \oplus \pi^*h^B$. Then*

$$(2\pi)^k\chi(M) = \int_M \mathrm{Pf}^g - \sum_{j=0}^{\frac{f-1}{2}} \tilde{c}\left(\frac{f-1}{2} - j\right)\int_B \left(\mathrm{Pf}(h^B)\int_{N/B} P_{j,f}\left(h^V\right)\right) - \int_{\partial M}\mathrm{TPf}(\nabla_1^h, \nabla_1^g)$$

where $\nabla_1^h = \varphi\nabla^h\varphi^{-1}\big|_{r=0}$ is described in (7.10) and ∇_1^g is the restriction of the extension of $\varphi\nabla^g\varphi^{-1}$ to $r=0$. The form $\mathrm{Pf}(h^B)$ is zero, by definition, when $\dim B$ is odd.

For horizontal variations of the metric the right hand side is given by Theorem 5.11.

Proposition 7.13. *The following holds*

$$\sum_{j=0}^{\frac{f-1}{2}} \tilde{c}\left(\frac{f-1}{2}-j\right)\int_B \left(\mathrm{Pf}(h^B)\int_{N/B} P_{j,f}\left(h^V\right)\right) = \int_{\{0\}\times N} \mathrm{TPf}(\varphi\nabla'\varphi^{-1},\varphi\nabla^h\varphi^{-1}) \tag{7.16}$$

where ∇' *is the connection from* (7.2).

Proof. The left hand side is equal to

$$\int_{\partial M} \mathrm{TPf}(\varphi\nabla^h\varphi^{-1}, s_h(0)) = \lim_{r\to 0}\int_{\{r\}\times N} \mathrm{TPf}(\varphi\nabla^h\varphi^{-1}, s_h(r)).$$

Clearly $s_h(0)=\partial_r$ and one sees easily from (7.10) that at $r=0$ the block diagonal components of $\varphi\nabla^h\varphi^{-1}$ with respect to $\mathbb{R}\oplus TN$ are in fact given by the connection

$$\varphi\nabla'\varphi^{-1} = d\oplus\pi_2^*\nabla^{VN}\oplus\pi_2^*\pi^*\nabla^B.$$

□

Remark 7.14. We used Theorem 1.2 to justify Proposition 7.13. But one can turn the tables around and give an alternative proof to Theorem 1.2 by providing a computational argument for (7.16).

We use this result in order to give a more geometric expression to the boundary contribution for first order perturbations of conical model metrics.

Definition 7.15. Let g be a first order perturbation of the metric

$$h = dr^2\oplus r^2 g^N$$

on $(-\epsilon,0]\times N$. Define the second fundamental form II^g of $\partial M := \{0\}\times N$ as follows:

$$\mathrm{II}^g(X,Y) := h^\varphi\left((\nabla_1^g)_X Y,\partial_r\right) = g^\varphi\left((\nabla_1^g)_X Y,\partial_r\right)$$

where $\nabla_1^g = \varphi\nabla^g\varphi^{-1}\big|_{TM}\big|_{\partial M}$ is the connection resulting from Theorem 7.8.

Denote by R^N the curvature form of the metric g^N and set

$$\mathcal{G}^{\partial M}_{j,2k-1} := \frac{1}{j!(2k-1-2j)!}\mathcal{B}_{h^\varphi}\left((R^N)^j\wedge(\mathrm{II}^g)^{2k-1-2j}\right).$$

Theorem 7.16. *For first order perturbations* g *of conical metrics* $dr^2\oplus r^2g^N$ *the following holds*

$$(2\pi)^k\chi(M) = \int_M \mathrm{Pf}^g - \sum_{j=0}^{k-1}(-1)^j(2j-1)!!\int_{\partial M}\mathcal{G}^{\partial M}_{k-1-j,2k-1}.$$

Proof. Proposition 7.13 and Theorem 7.12 together say that the contribution of the boundary is

$$\int_{\partial M}\mathrm{TPf}(\varphi\nabla'\varphi^{-1},\varphi\nabla^h\varphi^{-1}) + \int_{\partial M}\mathrm{TPf}(\varphi\nabla^h\varphi^{-1},\varphi\nabla^g\varphi^{-1}) = \int_{\partial M}\mathrm{TPf}(\varphi\nabla'\varphi^{-1},\varphi\nabla^g\varphi^{-1}).$$

In the conical case

$$\varphi\nabla'\varphi^{-1} = d\oplus\pi_2^*\nabla^N.$$

and these are also the block-diagonal components of $\varphi\nabla^g\varphi^{-1}$ at $r=0$. In order to justify this let us take another look at (7.13). When X is tangent to ∂M then since C^φ is the identity on ∂M we get that at $r=0$ one has

$$h^\varphi\left((\nabla_X^\varphi C^\varphi)(Y'),Z'\right) = 0.$$

On the other hand, $\varphi^{-1}(Y') = r^{-1}Y'$ and $\varphi(X) = rX$ for $X,Y'\in\Gamma(TN)$. Then the factors r^{-1} and r cancel each other out and one has:

$$h^\varphi\left((\nabla^\varphi_{\varphi^{-1}(Y')}C^\varphi)(\varphi(X))),Z'\right) = 0.$$

The last term of (7.13) is similar and therefore also vanishes. We conclude that for $X \in T\partial M$, $Y', Z' \in T\partial M$

$$h^{\varphi}(\omega^{\varphi}(X)(Y'), Z') = 0.$$

One sees easily that the same holds for $Y', Z' = \partial_r$. This justifies the claim that the block diagonal components of $\varphi\nabla^g\varphi^{-1}$ and $\varphi\nabla^h\varphi^{-1}$ when restricted to $r = 0$ are the same. But the block diagonal components of $\varphi\nabla^h\varphi^{-1}$ are the same as those of $\varphi\nabla'\varphi^{-1}$. Finally, the off-diagonal components of $\varphi\nabla^g\varphi^{-1}$ are exactly the components of II^g.

The situation is similar now to the proof of the Gauss-Bonnet formula 1.1 and $\mathrm{TPf}(\varphi\nabla'\varphi^{-1}, \varphi\nabla^g\varphi^{-1})$ can be computed accordingly. □

8. PERTURBATIONS OF MANIFOLDS WITH FIBERED BOUNDARY

Recall that an end of a manifold with fibered boundary is modeled on $(1, \infty) \times N$ with the metric

$$dr^2 \oplus g^V \oplus r^2\pi^* g^B$$

It is convenient to let $u = r^{-1}$ and then with the new coordinate the metric on $U^c = (0, 1) \times N$ is of type:

$$h = (d(u^{-1}))^2 \oplus g^V \oplus u^{-2}\pi^* g^B.$$

This suggests one should consider, in the spirit of the previous section, the following endomorphism $\varphi : \mathbb{R} \oplus TN\big|_U$:

$$\varphi(s, v, w) = (u^{-2}s, v, u^{-1}w), \qquad\qquad s \in \mathbb{R},\ v \in \pi_2^* VN,\ w \in \pi_2^*\pi^* TB.$$

where we use ∂_u as the coordinate on $\mathbb{R}$. Then clearly

$$h^{\varphi}(X', Y') := h(\varphi^{-1}(X'), \varphi^{-1}(Y'))$$

extends to a smooth metric on $(-1, 0] \times N =: U$.

Theorem 8.1. *Let ∇^h be the Levi-Civita connection of h on U^c. Then $\varphi\nabla^h\varphi^{-1}$ extends to a smooth connection on U which is metric with respect to h^{φ}.*

Proof. The structure of the proof is the same as that of Theorem 7.1. So we will only revise the main points.

The auxiliary connection ∇' is

$$\nabla' = \left[d - \frac{2}{u}du\right] \oplus \pi_2^*\nabla^{VN} \oplus \left[\left(\frac{\partial}{\partial u} - \frac{1}{u}\right)du + \pi_2^*\pi^*\nabla^B\right]$$

where d is the trivial connection on $\mathbb{R}$ and ∇^{VN} is the projection of the Levi-Civita connection of a slice $u = \mathit{const}$ to VN.

One notices easily that

(a) $\varphi\nabla'\varphi^{-1}$ extends smoothly;
(b) $d - \frac{2}{u}du$ and $\pi_2^*\nabla^{VN}$ are the projections of the Levi-Civita connection ∇^h to $\mathbb{R}$ and to $\pi_2^* VN$ respectively.
(c) ∂_u is orthogonal to the slices and the unit normal vector is $u^2\partial_u$; the vector field $X = u^2\partial_u$ satisfies the conditions of Lemma 3.1 and this allows the computation of the second fundamental form of the slices in the same vein we did before.

One then carefully analyzes the blocks of the 1-form $\varphi(\nabla^h - \nabla')\varphi^{-1}$ and sees that they extend as well. □

Remark 8.2. One might prefer to work directly with the r coordinate. In that case one first needs to turn $(1, \infty]$ into a manifold and this can be done via the unique chart $(-1, 0] \to (1, \infty]$ where $u \to -1/u$ for $u \neq 0$ and $0 \to \infty$. Then the vector field that trivializes the tangent bundle of $(1, \infty]$ (using the standard coordinate of $(1, \infty)$ is $\widetilde{\partial_r} := r^2\partial_r$ which makes sense also at ∞ and corresponds to ∂_u. Consequently the metric on $(1,\infty) \times N$ in these coordinates can be written as $r^4 \widetilde{dr}^2 \oplus g^V \oplus r^2 g^B$ and $\varphi(s,v,w) = (r^2 s, v, rw)$, etc.

Definition 8.3. A perturbation of h is a metric g such that g^φ extends smoothly to a metric on $TM\big|_U$ and

$$g^\varphi = h^\varphi + f(u)\alpha$$

for some smooth function f on $(-1,0]$ that vanishes at 0 and smooth, bilinear, symmetric form α on $TM\big|_U$. It is called of first, resp. second order if $f(u) = O(u)$, respectively $f(u) = O(u^2)$.

Lemma 8.4. *A perturbation of first, resp. second order for the metric* $h = dr^2 \oplus r^2 g^N$ *on* $(1,\infty)\times N$ *is a metric* g *such that*

$$g = h + O(r^{-1}) \cdot \gamma_N(r) dr^2 + O(1) \cdot (dr \otimes \beta_N(r) + \beta_N(r) \otimes dr) + O(r) \cdot \alpha_N(r)$$

respectively

$$g = h + O(r^{-2}) \cdot \gamma_N(r) dr^2 + O(r^{-1}) \cdot (dr \otimes \beta_N(r) + \beta_N(r) \otimes dr) + O(1) \cdot \alpha_N(r)$$

where $\gamma_N(r)$, $\beta_N(r) \in \Omega^1(N)$ *and* $\alpha_N(r) \in \Gamma^+(T^*N \otimes T^*N)$ *are smooth families of* 0 *and* 1*-forms, resp. symmetric* $(1,1)$ *double forms on* N *which extend smoothly at* ∞*, i.e. when composed with* $-1/u$ *they smoothly at* $u = 0$.

Proof. Straightforward. ☐

Example 8.5. Recall that a catenoid in $\mathbb{R}^3$ has the following parametrization

$$\mathcal{C} = \{(\cosh(v)\theta, v) \in \mathbb{R}^3 \mid \theta \in S^1,\ v \in \mathbb{R}\}.$$

Use the change of coordinates $v = \operatorname{arcsinh}(r)$ in order to write the metric as

$$dr^2 + (1+r^2)d\theta^2$$

where ∂_θ is the unit tangent vector on S^1 with the round metric. Clearly this is a second order perturbation of the flat metric $dr^2 + r^2 d\theta^2$.

Theorem 8.6. *For a first order perturbation* g *of* h *the connection* $\varphi\nabla^g\varphi^{-1}$ *extends to a smooth connection, while for a second order perturbation the restriction of* $\varphi\nabla^g\varphi^{-1}$ *to* $u = 0$ *(or* $r = \infty$*) coincides with the restriction of* $\varphi\nabla^h\varphi^{-1}$.

Proof. Almost identical to Theorem 7.8. Notice that in formula (7.13), $\nabla^{\varphi}_{\varphi^{-1}(Y')}C^{\varphi}$ makes sense at $u = 0$ as $\varphi^{-1}(s,v,w) = (u^2 s, v, uw)$ while ∇^φ and C^φ extend by Theorem 8.1 and Def. 8.3 respectively. ☐

Corollary 8.7. *The Gauss-Bonnet formula of Theorem 1.3 holds for second-order perturbations of a metric with fibered boundary.*

Example 8.8. For the catenoid, a minimal surface, the total Gaussian curvature is -4π, the Euler characteristic is 0, while each end contributes to the Gauss-Bonnet formula with 1 which is the integral of $(2\pi)^{-1} \cdot \mathrm{TPf}(S^1, g_{\mathrm{round}}, 1)$.

9. RIEMANNIAN ORBIFOLDS WITH SIMPLE SINGULARITIES

Let M be a Riemannian manifold and suppose G is a finite group that acts by isometries on M such that the following properties are satisfied:

(i) $\mathrm{Fix}_G(M)$ is a (necessarily closed) submanifold of M;
(ii) G acts freely on $M \setminus \mathrm{Fix}_G(M)$.

The quotient $\hat{M} := M/G$ is an example of a Riemannian orbifold. We use the following definition (see [6]):

Definition 9.1. A Riemannian orbifold $\hat{M}$ is a Hausdorff topological space endowed with a countable basis of open charts U_i, closed under finite intersection such that each chart U_i is homeomorphic with the quotient of an open set $\tilde{U}_i \subset \mathbb{R}^n$ endowed with a Riemannian metric g_i (that turns $\tilde{U}_i$ into a geodesically convex set) modulo the action of a finite group G_i that acts effectively by isometries on $\tilde{U}_i$. Moreover, the following data is part of the structure:

For each inclusion $U_i \subset U_j$ there exist

(i) an injective group morphism $\phi_{ij} : G_i \to G_j$;
(ii) an isometric embedding $\tilde{f}_{ij} : \tilde{U}_i \to \tilde{U}_j$, equivariant with respect to ϕ_{ij}

fitting a commutative diagram

$$\begin{tikzcd}
\tilde{U}_i \arrow[r, "\tilde{f}_{ij}"] \arrow[d] & \tilde{U}_j \arrow[d] \\
\tilde{U}_i/G_i \arrow[r, "\tilde{f}_{ij}/G_i"] & \tilde{U}_j/\phi_{ij}(G_i) \arrow[d] \\
 & \tilde{U}_j/G_j \\
U_i \arrow[uu, "\sim"] \arrow[r, hook, "i"] & U_j \arrow[u, "\sim"]
\end{tikzcd}$$

where $i : U_i \to U_j$ is the canonical inclusion.

Clearly, every open subset of an orbifold is an orbifold.

Definition 9.2. Let M and N be two Riemannian orbifolds. Then a homeomorphism $f : M \to N$ is an isometry if it is a local isometry, i.e., if for every pair $(m, n) \in \Gamma_f$ there exist

(a) charts $m \in U \subset M$, $n \in D \subset N$ with corresponding open sets $\tilde{U} \subset \mathbb{R}^n$ and $\tilde{D} \subset \mathbb{R}^n$ and groups G_U and G_D
(b) a group isomorphism $\phi : G_U \to G_D$ and
(c) an isometry $\tilde{f} : \tilde{U} \to \tilde{D}$ which is equivariant with respect to ϕ

such that the next diagram commutes

$$\begin{tikzcd}
\tilde{U} \arrow[r, "\tilde{f}"] \arrow[d] & \tilde{D} \arrow[d] \\
\tilde{U}/G_U \arrow[r, "\tilde{f}/G_U"] \arrow[d, "\sim"] & \tilde{D}/G_D \arrow[d, "\sim"] \\
U \arrow[r, "f"] & D
\end{tikzcd}$$

For every point $p \in M$, the isomorphism class of the isotropy group G_p is unambiguously defined. In a chart $U_i \ni p$ the group G_p is represented by the conjugacy class of the isotropy group of a lift $\tilde{p} \in \tilde{U}_i$.

Definition 9.3. The singular locus Z of an orbifold is:

$$Z := \{p \in M \mid G_p \neq \{e\}\}.$$

From the above definitions it is clear that $\hat{M} \setminus Z$ inherits a Riemannian manifold structure and we denote the metric by g. We will consider Riemannian orbifolds $\hat{M}$ for which the singular strata have a "nice" structure.

Definition 9.4. A Riemannian orbifold $\hat{M}$ is called with *simple singularities* if each connected component Z_i of Z has the property that there exists

- an open neighborhood D_i of Z_i,
- a finite group Γ_i and
- a Riemannian manifold M_i

such that

(i) Γ_i acts by isometries on M_i, $\mathrm{Fix}_{\Gamma_i}(M_i)$ is a compact submanifold in M_i and Γ_i acts freely on $M_i \setminus \mathrm{Fix}_{\Gamma_i}(M_i)$;

(ii) There exists an isometry of Riemannian orbifolds $h_i : D_i \to M_i/\Gamma_i$ such that

$$h_i(Z_i) = \mathrm{Fix}_{G_i}(M_i).$$

Any Riemannian orbifold with isolated singularities satisfies the previous definition. Denote by $\mathrm{Fix}(\hat{M})$ the set of connected components of the singular locus Z.

Theorem 9.5. *Let $\hat{M}$ be a compact Riemannian orbifold with simple singularities of dimension $2k$ and let g be the Riemannian metric on $\hat{M} \setminus Z$. Then*

$$\chi(\hat{M}) = \frac{1}{(2\pi)^k} \int_{\mathrm{Int}\,\hat{M}} \mathrm{Pf}^g - \sum_{Z_i \in \mathrm{Fix}(\hat{M})} \frac{\chi(Z_i)}{|G_i|}. \tag{9.1}$$

Proof. Fix a connected component $Z \in \mathrm{Fix}(\hat{M})$ and let D be the neighborhood of Z of Definition 9.4 such that $D \simeq M/\Gamma$. Let $B := \mathrm{Fix}_\Gamma(M)$. Since the action of Γ on M is via isometries in the induced action $\Gamma \times TM \to TM$ via differentials, the subset $S(\nu B)$ is invariant. Moreover, the action is free and linear in every fiber $S(\nu_b B)$.

Now let Γ act trivially on $(-\epsilon, 0]$. Then it is straightforward to see that

$$\exp : (-\epsilon, 0] \times S(\nu B) \to M, \qquad (r, p, v) \to \exp_p(rv)$$

is a Γ-equivariant map since every isometry $g \in \Gamma$ will take a geodesic with initial conditions (p, v) to a geodesic with initial conditions $(gp, d_p g(v))$.

It follows that one can find an (equivariant) tubular neighborhood for every $Z \in \mathrm{Fix}(\hat{M})$ whose boundary is a quotient $N = S(\nu B)/\Gamma$. One applies Gauss-Bonnet for manifolds with boundary in the complement of these tubular neighborhoods in $\hat{M}$ and then passes to limit $r \to 0$. So one can restrict attention to what happens in the neighborhood D with the limits of integrals of transgressions.

Recall now that the manifold $\tilde{M} := (-\epsilon, 0] \times S(\nu B)$ has a model degenerate metric and in fact Γ leaves invariant this model metric. For that it is enough to justify that Γ leaves invariant the splitting $TS(\nu B) = VS(\nu B) \oplus HS(\nu B)$, but that is obvious.

At this point, one analyzes first the case where the exponential map $\exp : D(\nu B) \to M$ is an isometry onto its image, in which case the induced map:

$$\exp/\Gamma : (D(\nu B)/G) \setminus \{0\} \to (M/G) \setminus Z$$

is an isometry onto its image where $\{0\}$ is the zero section of the disk bundle $D(\nu B)$. Use Examples 5.2 and 5.3 in order to conclude that formula (9.1) holds in this case since the integral in the fiber is an integral over $S(\nu_b B)/\Gamma$ with the sphere being endowed with the round metric.

In the general case, one notices that the degenerate metric induced on $D(\nu B)/G$ is a second order perturbation of the degenerate model metric because of Theorem 7.10. □

10. APPLICATIONS

Corollary 10.1. *Let $\hat{M}$ be a compact Riemannian orbifold with simple singularities of dimension $2k$ and let g be the Riemannian metric on $\hat{M} \setminus Z$. Then $\frac{1}{(2\pi)^k} \int_{\mathrm{Int}\,\hat{M}} \mathrm{Pf}^g$ is rational.*

This follows immediately from theorem 9.5. If the orbifold $\hat{M}$ is the finite quotient of a closed smooth manifold X, one can obtain this result from the Gauss-Bonnet formula on X, however such a X does not exist in general.

The Gauss-Bonnet formulæ proved here imply some global obstructions for the existence of flat cobordisms with prescribed ends of fibered boundary- or incomplete edge type.

The simplest instance of such an obstruction arises for even-dimensional cones modeled by quotients of the round sphere, for instance lens spaces.

Corollary 10.2. *There do not exist flat metrics on a compact manifold with a cone singularity modeled on $\Gamma\backslash S^{2k-1}$ for a nontrivial group of isometries Γ acting freely on the round sphere.*

Proof. When we remove a point from a smooth manifold M, the Euler characteristic decreases by 1, and this is reflected in the transgression form of Theorem 4.6 on the odd round sphere: the integral of this local transgression form must equal 1 (Remark 4.8). We deduce that on the quotient of S^{2k-1} by a finite group of isometries Γ acting freely, this transgression form integrates to $1/|\Gamma|$. The Pfaffian form of a flat metric vanishes, hence $1/|\Gamma| \in \mathbb{Z}$, thus Γ must be trivial. ☐

More generally, for edge metrics Theorems 1.2 and 7.11 imply some restrictions for the existence of a flat manifold (M, g) bounding an edge singularity modeled on a fibration $N \to B$ with fibers of constant curvature 1. Each fiber is isometric to the quotient of the round sphere by the free action of a finite group Γ of isometries of S^{2f-1}, hence the trangression form on each fiber is constant equal to $1/|\Gamma|$.

It follows immediately form the Gauss-Bonet formula for second-order perturbation of model edge singularities (Theorem 7.11) that in this context, the order of Γ must divide $\chi(B)$.

Finally, Theorem 1.3 implies an obstruction for the existence of flat manifolds with fibered boundary ends. Assume that the metric near the fibered boundary is modeled by a fibration $N \to B$ where B is a the quotient of the round sphere by the free action of a finite group Γ of isometries of S^{2b-1}. The Gauss-Bonnet formula for second-order perturbations of fibered boundary metrics shows that the order of Γ must divide $\chi(F)$.

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