

# BOUNDARIES OF LOCALLY CONFORMALLY FLAT MANIFOLDS IN DIMENSIONS $4k$

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ABSTRACT. We give global restrictions on the possible boundaries of compact, orientable, locally conformally flat manifolds of dimension  $4k$  in terms of integrality of eta invariants.

## 1. INTRODUCTION AND STATEMENT

There exist obstructions for  $4k - 1$ -manifolds bounding a compact  $4k$ -dimensional Riemannian manifold with special geometry.

- Chern and Simons [6] showed that the Chern-Simons invariant of a closed oriented 3-manifold conformally immersed in  $\mathbb{R}^4$  must vanish modulo  $\mathbb{Z}$ .
- Atiyah, Patodi and Singer [2] proved that the eta invariant of the odd signature operator on a closed 3-manifold conformally embedded in  $\mathbb{R}^4$  must vanish.
- Long and Reid [16] considered a possibly noncompact hyperbolic 4-manifold with totally geodesic boundary. They allowed non-compact cuspidal ends, and cut out the cusp by a flat totally umbilic section. They obtained a compact hyperbolic manifold whose boundary components are either totally geodesic or umbilic and flat. They proved that the eta invariant of the boundary must be an even integer.
- Xianzhe Dai [8] showed that the eta invariant of a locally conformally flat manifold  $(M, h)$  vanishes modulo  $2\mathbb{Z}$  whenever  $M$  is the oriented, totally umbilic boundary of a locally conformally flat compact manifold  $(X, g)$ . Dai's result contains Long and Reid's as a particular case.

With the notable exception of [6], the essential ingredient in the proof of the above results is the Atiyah-Patodi-Singer signature formula for manifolds with boundary equipped with metrics of product type near the boundary [1]. In dimension 4, this formula reads for instance:

$$(1) \quad -\frac{1}{24\pi^2} \int_X \text{tr}(R^2) - \frac{1}{2}\eta(M, h) = \text{signature}(X) \in \mathbb{Z}.$$

Our main result here is:

**Theorem 1.** *Let  $(X, g)$  be a compact oriented locally conformally flat Riemannian manifold of dimension  $4k$ , with smooth boundary  $(M, h)$ . Then the eta invariant of the odd signature operator of  $(M, h)$ , and also the eta invariant of the Dirac operator if  $M$  has a spin structure, belong to  $2\mathbb{Z}$ .*

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*Date:* September 12, 2016.

Partially supported by the CNCS project PN-II-RU-TE-2012-3-0492.

Compared to the corresponding results of [2] and [8], we do not ask the interior metric  $g$  to be flat (but only locally conformally flat), respectively we do not impose any restrictions on the second fundamental form other than the usual Codazzi-Mainardi equation.

The proof is algebraic in nature, and relies on the analysis of a transgression form for the Pontryagin forms appearing in the Atiyah-Patodi-Singer index formula.

In dimension 4, Hitchin [14] showed that for a compact manifold with boundary whose metric is conformally equivalent to an Einstein metric, the transgression of the first Pontryagin form vanishes. Thus our Theorem 3 could be viewed as a partial generalization of Hitchin's result to higher dimensions.

**Acknowledgements.** I am grateful to the referee for several useful comments about boundary-value problems, and for pointing out valuable bibliographical references.

## 2. THE INDEX FORMULA FOR PRODUCT METRICS

Let us recall the index formula of Atiyah, Patodi and Singer [1]. Let  $(X, g)$  be a Riemannian metric on a oriented compact manifold  $X$ , with boundary  $M$ , and denote by  $h$  the induced metric on  $M$ . The metric  $g$  is said to be of *product type* if there exists an embedding of manifolds with boundary  $[0, \epsilon) \times M \hookrightarrow X$  such that the pull-back of the metric  $g$  takes the form

$$g = dt^2 + h.$$

Intrinsically, this condition can be reformulated as

$$L_\nu g = 0 \text{ near } M,$$

where  $L$  denotes Lie derivative and  $\nu$  is the geodesic unit vector field normal to the boundary.

**Theorem 2** ([1]). *Let  $(X, g)$  be a compact Riemannian manifold with boundary, and assume that the metric is of product type near the boundary  $(M, h)$ . Let  $(D, A)$  denote one of the following pairs of elliptic operators on  $X$ , respectively on  $M$ :*

- (1)  *$D$  is the signature operator on  $X$ , and  $A$  is the odd signature operator on  $M$ ;*
- (2) *When  $X$  has a fixed spin structure with spinor bundle  $S$ ,  $D$  is the Dirac operator on  $(X, S)$ , and  $A$  is the Dirac operator on  $M$  for the induced spin structure.*

*Then the index of  $D^+$  (with the non-local APS boundary condition defined by  $A$ ) equals*

$$(2) \quad \mathbb{Z} \ni \text{index}(D^+) = \int_X \exp \left[ \frac{1}{2} \text{tr} \log \left( P \left( \frac{R}{2\pi i} \right) \right) \right] - \frac{1}{2} \eta(A).$$

*where  $R$  is the Riemannian curvature tensor of  $g$ , and  $P$  is a specific Taylor series:*

$$P(x) = \begin{cases} \frac{x}{\tanh(x)} & \text{for the signature operator;} \\ \frac{x/2}{\sinh(x/2)} & \text{for the Dirac operator.} \end{cases}$$

Only the monomials of degree at most  $2k$  in the series  $P$  contribute to the integrand.

The reader unfamiliar with the definition of the index, the signature and the Dirac operator, the chiral operator  $D^+$ , the eta invariant, the non-local APS boundary condition, and the proof of the index formula is referred to the original paper of Atiyah, Patodi and

Singer [1]. We review them in section A for metrics which are not necessarily of product type. But in fact, none of these ingredients really matter in the present work. Indeed, as a consequence of the index formula, the eta invariant of  $A$  on  $M$ , defined in [1] as a spectral invariant intrinsic to  $M$  and the operator  $A$ , can be computed modulo  $\mathbb{Z}$  using the index formula on some oriented manifold  $X$  bounding  $M$  (if any) with metric of product type:

$$(3) \quad \frac{1}{2}\eta(A) \equiv \int_X e^{\frac{1}{2}\text{tr} \log P(\frac{R}{2\pi i})} \pmod{\mathbb{Z}}.$$

Reversing the logical order of thought, assuming that  $M$  is the boundary of some oriented  $X$  we could *define* the eta invariant modulo  $2\mathbb{Z}$  by this equation. The independence of the right-hand side on  $(X, g)$  (with  $g$  of product type) is a consequence of Hirzebruch's signature formula, respectively of the Atiyah-Singer index theorem.

If  $g$  is locally conformally flat, the Weyl curvature tensor vanishes, and so the integrand in (3), which only depends on the Weyl tensor, vanishes in positive degrees. It follows that for such metrics on  $X$  the eta invariant of  $A$  is an even integer, under the condition that  $g$  is of product type. But theorem 1, our main result, proved in section 5, shows that this last hypothesis is not necessary!

As a corollary of our proof of theorem 1, we obtain an algebraic extension of the index formula to certain metrics which are not of product type. The precise formulation of the boundary-value problem in this non-product case is recalled in Appendix A.

**Theorem 3.** *The index formula (2) is valid for metrics  $g$  which are not necessarily of product type near  $M$ , but are locally conformally flat to the first order near  $M$ , in the sense that the Weyl tensor of  $g$  vanishes at every  $p \in M$ .*

### 3. TRANSGRESSION FORMS

The Atiyah-Patodi-Singer index formula can be extended to metrics which are not of product type, by adding a boundary correction term [10]. The rough nature of this term is well-understood: it is a polynomial in the second fundamental form, its twisted exterior derivative, and the curvature tensor of the boundary. In dimension  $4k = 4$  the correction term is explicitly given in [9]. This boundary term vanishes when the boundary is totally geodesic, or even just umbilic. We want to prove that this transgression form vanishes under the hypothesis of theorem 3. Proposition 6 is quite general and also well-known, it follows for instance as a particular case of the results from [11], but we are obliged to include the proof below since it forms the starting point of our computations.

Let

$$[0, \epsilon) \times M \hookrightarrow X, \quad (t, p) \mapsto \exp_p(t\nu)$$

be the embedding defined by the geodesic normal flow from the boundary, where  $\nu$  is the unit inner normal field. The pull-back of the metric  $g$  takes the form

$$g = dt^2 + h(t)$$

where  $h(t)$  is a smooth family of metrics on  $M$  starting at  $h(0) = h$ . We can then compute

$$L_{\partial_t} g = \partial_t h(t) \text{ near } M,$$

Let  $g_0$  be any metric on  $X$  which equals  $dt^2 + h$  near  $M$ . Clearly  $g_0$  is of product type, and shares with  $g$  the same geodesics normal to the boundary. Let  $\nabla^1, \nabla^0$  denote the Levi-Civita covariant derivatives with respect to  $g, g_0$ , and define

$$\theta := \nabla^1 - \nabla^0 \in \Lambda^1(X, \text{End}(TX)).$$

Let also  $\nabla^M$  denote the Levi-Civita connection of  $(M, h)$ . The second fundamental form  $\mathbb{I}$  of the inclusion  $(M, h) \hookrightarrow (X, g)$  is defined for vector fields  $U, V$  tangent to  $M$  by

$$\nabla_U^1 V = \nabla_U^M V + \mathbb{I}(U, V)\partial_t.$$

Notice that  $\nabla_U^M V = \nabla_U^0 V$  because  $g^0$  is of product type. Let  $W$  be the Weingarten operator,  $h(WU, V) = \mathbb{I}(U, V)$ . Then for vectors  $U, V$  tangent to  $M$  we have:

$$(4) \quad \theta(U)V = \mathbb{I}(U, V)\partial_t, \quad \theta(U)\partial_t = -W(U), \quad \theta(\partial_t) = -W.$$

The last identity holds because  $\partial_t$  is a geodesic vector field both for  $g$  and  $g^0$ .

Consider the segment of connections

$$\nabla^s = \nabla^0 + s\theta$$

linking  $\nabla^0$  to  $\nabla^1$  for  $s \in [0, 1]$ , and let  $R^s$  be the curvature tensor of  $\nabla^s$  (so  $R^1$  is the curvature of the initial metric  $g$ ). We have

$$(5) \quad R^s = R^0 + sd^{\nabla^0}\theta + s^2\theta^2 \in \Lambda^*(X, \text{End}(TX))[s].$$

Here we pause to explain the notation. Whenever  $\omega, \omega' \in \Lambda^*(X)$  and  $B, B' \in \text{End}(TX)$ , we define the product of the endomorphism-valued forms  $\omega \otimes B$  and  $\omega' \otimes B'$  as follows:

$$\omega \otimes B \cdot \omega' \otimes B' := \omega \wedge \omega' \otimes BB'.$$

Thus  $\theta^2$  is an endomorphism-valued 2-form. The bracket  $[s]$  signifies that the dependence of  $R^s$  on  $s$  is polynomial (of degree 2). For later use, define the trace:

$$\text{tr}(\omega \otimes B) := \text{tr}(B)\omega \in \Lambda^*(X).$$

We note the obvious trace identity:

$$(6) \quad \text{tr}(\omega \otimes B \cdot \omega' \otimes B') = (-1)^{\deg(\omega) \cdot \deg(\omega')} \text{tr}(\omega' \otimes B' \cdot \omega \otimes B).$$

**Lemma 4.** *Let  $\{S_j\}_{1 \leq j \leq 4k-1}$  be a local orthonormal basis for  $TM$  consisting of eigenvectors of  $W$ , namely  $WS_j = \lambda_j S_j$  for  $\lambda_j \in \mathbb{R}$ . Then on the boundary  $M$  we have*

$$i_M^* \theta^2 = \sum_{i \neq j} \lambda_i \lambda_j S^i \wedge S^j \otimes [S^i \otimes S_j] = \sum_{i < j} \lambda_i \lambda_j S^i \wedge S^j \otimes [S^i \otimes S_j - S^j \otimes S_i].$$

*Proof.* Evident from (4). □

From this lemma, we compute immediately  $i_M^* \theta^3 = 0$  on  $M$ .

The following proposition is standard:

**Proposition 5.** *Let  $Q$  be a polynomial. Then*

$$\text{tr}(Q(R^1)) - \text{tr}(Q(R^0)) = d \int_0^1 \text{tr}(\theta Q'(R^s)) ds.$$

Here  $Q(R^s)$  is an endomorphism-valued form as explained above.

*Proof.* Since  $\frac{d}{ds}\nabla^s = \theta$ , we have  $\frac{d}{ds}R^s = d^{\nabla^s}\theta$ . Therefore, using the trace identity,

$$\frac{d}{ds}\text{tr}(Q(R^s)) = \text{tr}(d^{\nabla^s}(\theta)Q'(R^s)).$$

By the second Bianchi identity,  $d^{\nabla^s}R^s = 0$ , so  $d^{\nabla^s}Q'(R^s) = 0$  and therefore

$$d^{\nabla^s}(\theta)Q'(R^s) = d^{\nabla^s}(\theta Q'(R^s)).$$

To conclude, use the identity  $\text{tr}(d^{\nabla}\cdot) = d\text{tr}(\cdot)$  and integrate from 0 to 1.  $\square$

The trace  $\text{tr}(Q(R^s))$  is a differential form depending polynomially on  $s$ . Its exponential is well-defined, and is again a polynomial in  $s$  with coefficients in  $\Lambda^*(X)$ . Moreover we have:

**Proposition 6.** *Let  $Q$  be a polynomial. Then*

$$e^{\text{tr}(Q(R^1))} - e^{\text{tr}(Q(R^0))} = d \int_0^1 \text{tr}(\theta Q'(R^s)) e^{\text{tr}(Q(R^s))} ds.$$

The proof goes exactly like in the preceding proposition. We use the fact that  $\text{tr}(Q(R^s))$  is an even form and the trace identity to obtain

$$\frac{d}{ds} \exp(\text{tr}(Q(R^s))) = d\text{tr}(\theta Q'(R^s)) \exp(\text{tr}(Q(R^s))).$$

#### 4. TRANSGRESSION FOR LOCALLY CONFORMALLY FLAT METRICS ON $X$

The *Schouten tensor* of the metric  $g$  is defined by

$$\mathfrak{Sch} = \frac{1}{n-2} \left( \mathfrak{Ric} - \frac{\text{scal}}{2(n-1)} \right),$$

where  $n = \dim(X) = 4k$ . We henceforth assume that the Weyl component of the curvature tensor of  $g$  vanishes at a point  $p \in M$ . At such  $p$  we then have [5, 1.116]

$$R^1 = \mathfrak{Sch} \otimes g,$$

meaning that

$$(7) \quad (\mathfrak{Sch} \otimes g)(U_1, U_2, U_3, U_4) = \mathfrak{Sch}(U_1, U_4)\langle U_2, U_3 \rangle + \mathfrak{Sch}(U_2, U_3)\langle U_1, U_4 \rangle \\ - \mathfrak{Sch}(U_1, U_3)\langle U_2, U_4 \rangle - \mathfrak{Sch}(U_2, U_4)\langle U_1, U_3 \rangle.$$

In particular, we deduce the following:

**Lemma 7.** *Assume that the Weyl tensor of  $g$  vanishes at  $p \in M$ . Then for mutually orthogonal vectors  $U_1, U_2, U_3, U_4 \in T_p X$  we have  $\langle R_{U_1 U_2}^1 U_3, U_4 \rangle = 0$ .*

*Proof.* If the vectors  $U_j$ 's are mutually orthogonal, all the scalar products in (7) vanish.  $\square$

From  $g$  we have constructed a product type metric  $g_0$ , and we have denoted by  $\theta$  the difference between the covariant derivatives corresponding to  $g$  and to  $g_0$ . Let  $\iota_M : M \hookrightarrow X$  denote the inclusion map. The main result of this section is the following vanishing result:

**Proposition 8.** *If the Weyl tensor of  $(X, g)$  vanishes at  $p \in M$ , then for every polynomial  $Q$ , the pull-back*

$$i_M^* \text{tr}(\theta Q(R^s)) \in \Lambda^*(M)$$

*vanishes at  $p$ .*

*Proof.* Let  $\{S_j\}_{j=1}^{4k-1}$  be an orthonormal basis of  $T_p M$  consisting of eigenvectors for the Weingarten operator  $W$ . We have

$$(8) \quad i_M^* \theta = \sum_{i=1}^{4k-1} \lambda_i S^i \otimes [S^i \otimes \partial_t - dt \otimes S_i],$$

$$i_M^* R^0 = R^M,$$

$$(9) \quad \langle d^{\nabla^0} \theta(S_i, S_j) S_h, \partial_t \rangle = \langle d^{\nabla^0} W(S_i, S_j), S_h \rangle = \langle R_{S_i S_j}^1 S_h, \partial_t \rangle =: R_{ijh\nu}^1.$$

Also from (8), using the fact that  $\partial_t$  is parallel with respect to  $g_0$ ,

$$(10) \quad \langle d^{\nabla^0} \theta(S_i, S_j) S_h, S_l \rangle = 0.$$

Equation (9) is just the Codazzi-Mainardi constraint for the isometric embedding  $(M, h) \subset (X, g)$ . Since  $\partial_t \perp M$ , from (9), (10) and lemma 7 we deduce

$$(11) \quad i_M^* d^{\nabla^0} \theta = \sum_{a < b} S^a \wedge S^b \otimes [R_{abb\nu}^1 (S^b \otimes \partial_t - dt \otimes S_b) + R_{abav}^1 (S^a \otimes \partial_t - dt \otimes S_a)].$$

**Lemma 9.** *Let  $1 \leq i, j, h, l \leq 4k - 1$  be four distinct indices, and assume that the Weyl tensor of  $g$  vanishes at  $p \in M$ . Then  $\langle R_{S_i S_j}^0 S_h, S_l \rangle = 0$ .*

*Proof.* From equation (5) with  $s = 1$ , Lemma 4 becomes

$$\langle R_{S_i S_j}^0 S_h, S_l \rangle + \langle d^{\nabla^0} \theta(S_i, S_j) S_h, S_l \rangle + \langle \theta^2(S_i, S_j) S_h, S_l \rangle = 0.$$

But the last two terms vanish by (11) and Lemma 7.  $\square$

By linearity, we may assume that  $Q$  is a monomial,  $Q(x) = x^l$ . Notice that  $i_M^* \theta$  and  $i_M^* R^s$  are skew-adjoint endomorphism-valued forms, the latter being also of even degree. If  $l$  is even, by taking adjoints it follows that the trace of  $i_M^* \theta (R^s)^l$  vanishes. Assume now that  $l$  is odd. We will show below that the diagonal of the form-valued endomorphism

$$i_M^* \theta (R^s)^l = i_M^* \theta (R^0 + s d^{\nabla^0} \theta + s^2 \theta^2)^l \in \Lambda^{2l+1}(M, \text{gl}(n))$$

written in the basis  $\{\partial_t, S_1, \dots, S_{4k-1}\}$  consists of 0's. Rewrite (4) as

$$(12) \quad i_M^* \theta = \sum_{i=1}^{4k-1} \lambda_i S^i \otimes [S^i \otimes \partial_t - dt \otimes S_i],$$

and decompose  $i_M^* \theta$  into

$$i_M^* \theta = \theta' - \theta'', \quad \theta' := \sum_{i=1}^{4k-1} \lambda_i S^i \otimes [S_i \otimes dt], \quad \theta'' := \sum_{i=1}^{4k-1} \lambda_i S^i \otimes [\partial_t \otimes S^i].$$

Since  $\theta''$  is the transpose of  $\theta'$  and  $R^s$  is skew-symmetric, it follows that

$$i_M^* \text{tr}(\theta(R^0 + sd^{\nabla^0} \theta + s^2 \theta^2)^l) = 2 \text{tr}(\theta' i_M^*(R^0 + sd^{\nabla^0} \theta + s^2 \theta^2)^l).$$

Introduce the following unified notation for the 2-forms entering in the expression of  $i_M^* R^s$ :

$$A^1 = i_M^* R^0 = R^M, \quad A^2 = i_M^* d^{\nabla^0} \theta, \quad A^3 = i_M^*(\theta \cdot \theta).$$

**Lemma 10.** *For every  $l \geq 1$ , and  $\alpha_1, \dots, \alpha_l \in \{1, 2, 3\}$ , the  $2l + 1$ -form*

$$\theta' A^{\alpha(1)} \dots A^{\alpha(l)}$$

can be written as

$$\sum_{i,j=1}^{4k-1} \sum_I C_{i,j,I} S^I \otimes [S_i \otimes S^j]$$

where  $I$  is a multi-index of length  $2l + 1$ , and the coefficient  $C_{i,j,I} \in C^\infty(M)$  equals zero unless  $i, j \in I$  and  $i \neq j$ .

*Proof.* Remark that  $\theta' A^\alpha = 0$  unless  $\alpha = 2$ , because the endomorphism component of  $\theta'$  restricted to  $T_p M$  vanishes, while  $A^1$  and  $A^3$  map  $T_p X$  into  $T_p M$ . Thus we can assume that  $\alpha(1) = 2$ . The form  $\theta' i_M^* d^{\nabla^0} \theta$  can be written, using (12) and (11), as

$$\sum_{I_1, I_2, I_3=1}^{4k-1} C_I S^I \otimes [S_{I_1} \otimes S^{I_3}].$$

In the above sum, the wedge product  $S^I = S^{I_1} \wedge S^{I_2} \wedge S^{I_3}$  clearly vanishes unless the three indices are mutually distinct, so we retain only the terms with  $I_1 \neq I_3$ . Once we established this initial step, the proof proceeds by induction. Assume that the conclusion of the lemma holds for  $l$ . We want to prove it for the product with one additional factor  $A^{\alpha(l+1)}$ . We claim that under the hypotheses  $i, j \in I$ ,  $I \neq j$ , the product

$$(13) \quad S^I \otimes [S_i \otimes S^j] \cdot A^{\alpha(l+1)}$$

vanishes for  $\alpha(l+1) = 2$ . Indeed, using (11), we see that in order to have a non-zero term in the product

$$S^I \otimes [S_i \otimes S^j] \cdot S^a \wedge S^b \otimes [R_{abb\nu}^1 (S^b \otimes \partial_t - dt \otimes S_b) + R_{abav}^1 (S^a \otimes \partial_t - dt \otimes S_a)],$$

we should have either  $j = a$  or  $j = b$ . Since by the induction hypothesis  $j \in I$ , it follows that in such a case the exterior product  $S^I \wedge S^a \wedge S^b$  vanishes.

Using Lemma 4, exactly the same argument as above shows the vanishing of the product (13) in the case where  $\alpha(l+1) = 3$ , i.e.,  $A^{\alpha(l+1)} = A^3 = \theta \cdot \theta$ .

In the remaining case  $\alpha(l+1) = 1$  we multiply  $S^I \otimes [S_i \otimes S^j]$  to the right by  $A^1 = R^M$ . By Lemma 9,

$$R^M = \sum_{a,b,c=1}^{4k-1} S^a \wedge S^b \otimes [R_{abac}^0 (S_a \otimes S^c - S_c \otimes S^a)].$$

The term corresponding to a triple  $(a, b, c)$  in the product  $S^I \otimes [S_i \otimes S^j] \cdot R^M$  is non-zero only if  $j = a$  or  $j = c$ . In the first case, the wedge product  $S^I \wedge S^a$  vanishes since by induction  $j \in I$ . Thus the result of the product (13) consists of terms of the form

$S^l \wedge S^a \wedge S^b \otimes [S_i \otimes S^a]$ . By the induction hypothesis  $i \in I$ . If  $a = i$ , again the wedge product vanishes, so we may retain only the terms with  $a \neq i$ . Evidently  $a \in I \cup \{a\}$ , proving the induction step.  $\square$

This lemma implies that for  $l \geq 1$ , the endomorphism-valued differential form  $\theta' i_M^*(R^s)^l$  is off-diagonal. For  $l = 0$  the same holds due to (12). Its trace thus vanishes, and since  $i_M^* \theta$  is the skew-symmetric component of  $2\theta'$  and  $l$  is odd, we conclude that  $i_M^* \text{tr}(\theta(R^s)^l) = 0$  as claimed.  $\square$

## 5. CONCLUSION

Let us derive the proofs of the statements announced in the beginning of the paper. Let  $g$  be a metric on  $X$  which is not necessarily of product type. Let  $(D, A)$  be one of the pairs of elliptic operators from the statement of theorem 2 corresponding to  $g$  and the induced metric  $h$  on  $M$ . Let  $g_0$  be the product type metric constructed from  $g$  in the beginning of section 3 by using the normal geodesic flow from the boundary, and  $D_0$  the corresponding operator on  $X$ .

From the Atiyah-Patodi-Singer index theorem for the product-type metric  $g_0$ , we write

$$(14) \quad \text{index}(D_0^+) = \int_X \exp \left[ \frac{1}{2} \text{tr} \log(P(R^0/2\pi i)) \right] - \frac{1}{2} \eta(A).$$

Define a power series  $Q(x) := \frac{1}{2} \log(P(x/2\pi i))$  (only the truncation up to degree  $4k$  matters here). From proposition 6

$$e^{\text{tr}Q(R^1)} - e^{\text{tr}Q(R^0)} = d \int_0^1 \text{tr}(\theta Q'(R^s)) e^{\text{tr}Q(R^s)} ds$$

hence by the Stokes formula,

$$(15) \quad \int_X e^{\text{tr}Q(R^1)} - e^{\text{tr}Q(R^0)} = \int_0^1 \left( \int_M \text{tr}(\theta Q'(R^s)) e^{\text{tr}Q(R^s)} \right) ds.$$

**Lemma 11.** *If the Weyl tensor of  $g$  vanishes at every  $p \in M$ , then*

$$\text{index}(D_0^+) = \int_X \exp \left[ \frac{1}{2} \text{tr} \log(P(R^1/2\pi i)) \right] - \frac{1}{2} \eta(A).$$

*Proof.* By proposition 8, the hypothesis ensures that the right-hand side of (15) vanishes, so the lemma follows from the index formula (14).  $\square$

*Proof of theorem 1.* We use now the hypothesis that  $g$  is locally conformally flat, i.e., the Weyl tensor of  $g$  vanishes on  $X$ . On the one hand, lemma 11 applies since the Weyl tensor in particular vanishes at every point  $p \in M$ . On the other hand, the Pontriagin form  $\text{tr} \log(P(R^1/2\pi i))$  vanishes identically, since in general it only depends on the Weyl tensor. We deduce

$$\frac{\eta(A)}{2} = -\text{index}(D_0^+) \in \mathbb{Z}. \quad \square$$



*Proof of theorem 3.* The index formula from lemma 11 applies since we assume the Weyl tensor of  $g$  to vanish at  $M$ . In Appendix A we recall the definition of the Fredholm index of  $D^+$  with the non-local spectral boundary condition defined by  $A$ , and prove by deformation that  $\text{index}(D_0^+) = \text{index}(D^+)$ , thereby ending the proof.  $\square$

## 6. CONSEQUENCES

The eta invariant of the odd signature operator is known for various classes of 3-manifolds, including lens spaces [2], [15] and hyperbolic manifolds [7]. These manifolds are quotients of the standard 3-sphere, respectively of the hyperbolic 3-space, by some discrete group of isometries, so they can be locally embedded isometrically in  $\mathbb{R}^4$ . By a result of Hirsch [13], every oriented compact 3-manifold can be  $C^\infty$  immersed in  $R^4$ , and is moreover the boundary of some oriented compact  $X$  from Thom [18]. Theorem 1 implies that a closed oriented 3-manifold cannot be isometric to the boundary of a locally conformally flat manifold, unless the sum of the eta invariants of its connected components is an even integer.

For instance, the lens space  $L(3, 1)$  is not the boundary of a locally conformally flat 4-manifold since its eta invariant is  $-\frac{1}{3}$ . Eleven out of the 12 closed hyperbolic 3-manifolds from [7, Table 2] have non-integral eta invariant, so they do not bound locally conformally flat 4-manifolds.

The eta invariant of the Dirac operator on Bieberbach (i.e., flat and closed) oriented 3-manifolds was computed by Pfäffle [17]. It is a topological invariant of the fundamental group and the spin structure. He found that for the manifolds  $G_j$ ,  $j \in \{2, 3, 4, 5\}$  from the Hantsche-Wendt classification of Bieberbach groups in dimension 3, there exist spin structures  $\sigma_j$  for which the eta invariant modulo 2 equals  $-\frac{2}{k}$ , where  $k \geq 2$  is the order of the holonomy group of  $G_j$  [17, Theorem 5.6]. The spin cobordism group is trivial in dimension 3, so for each  $j$  there exists a spin manifold  $(X_j, \tau_j)$  with spin boundary  $(G_j, \sigma_j)$ . Theorem 1 asserts that  $X_j$  cannot carry a locally conformally flat metric extending any of the flat metrics on  $G_j$ .

## APPENDIX A. THE INDEX PROBLEM FOR NON-PRODUCT METRICS

Let  $g$  denote an arbitrary smooth metric on  $X$  restricting to  $h$  on  $M$ . The geodesic flow with respect to  $g$  in the direction of the inner unit vector field normal to the boundary  $M$  gives, by the Gauss Lemma, a generalized cylinder decomposition (see [4]) of  $X$  near  $M$ :

$$g = dt^2 + h_t,$$

where  $\{h_t\}_{0 \leq t < \epsilon}$  is a smooth family of metrics on  $M$  with  $h_0 = h$ . Assume that either  $\dim(X)$  is a multiple of 4, or that  $X$  is even-dimensional and spin, and let  $D$  be the signature, respectively the spin Dirac operator. To unify notation, we denote in both cases by  $S$  the exterior bundle, respectively the spinor bundle on  $X$ . The operator  $D$  is symmetric, and odd with respect to the splitting  $S = S^+ \oplus S^-$  induced by the Hodge star, respectively by Clifford multiplication with the volume form:

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix} : C^\infty(X, S^+ \oplus S^-) \longrightarrow C^\infty(X, S^+ \oplus S^-).$$

The operator  $D^+ : C^\infty(X, S^+) \rightarrow C^\infty(X, S^-)$  is called the *chiral* component of  $D$ . Parallel transport in the bundle  $S$  yields a decomposition of  $D^+$  for small  $t$ :

$$(16) \quad D^+ = \sigma \cdot [\partial_t + A + tB_1(t) + B_0(t)]$$

where  $\sigma$  is an invertible bundle morphism, and the correction term  $B_j(t)$ ,  $j \in \{0, 1\}$ , is a family of differential operators on  $M$  of order  $j$  depending smoothly on  $t \geq 0$ . The self-adjoint operator  $A$  is the odd signature operator, respectively the spin Dirac operator on  $M$  with respect to the induced metric  $h$ , exactly like in the case where  $g$  is of product type near  $\{t = 0\}$ . The correction terms  $B_0, B_1$  vanish identically when the metric is of product type.

Let  $\Pi_{[0, \infty)}$  denote the spectral projector onto the non-negative eigenmodes of  $A$ :

$$\Pi_{[0, \infty)} = \frac{1}{2} \left( 1 + A^{-1}(A^2)^{\frac{1}{2}} \right)$$

(by definition,  $\Pi_{[0, \infty)}$  is the identity on the null-space of  $A$ ). This is a pseudodifferential operator of order 0. Consider the Atiyah-Patodi-Singer restriction of the operator  $D^+$  with respect to the spectral boundary condition defined by  $A$ :

$$D^+ : \{\phi \in C^\infty(X, S^+); \Pi_{[0, \infty)}(\phi|_M) = 0\} \rightarrow C^\infty(X, S^-)$$

and its closure in the Sobolev spaces  $H^1(X, S)$ , respectively  $L^2(X, S)$ :

$$D^+ : \{\phi \in H^1(X, S^+); \Pi_{[0, \infty)}(\phi|_M) = 0\} \rightarrow L^2(X, S^-).$$

Elliptic operators of this type have been studied beginning with [1] in the product case, where they are shown in particular to be Fredholm. In the general case, the Fredholm property is due to Grubb [12]. We also refer the reader to [3] for a self-contained analysis of Fredholm extensions of operators of the type (16).

As in section 3, let  $g_0$  be a metric on  $X$  of product type near  $M$ ,  $g_0 = dt^2 + h$  near  $\{t = 0\}$ , where  $h$  is the restriction of  $g$  to  $M$ . Implicitly we assume that  $g$  and  $g_0$  have the same geodesics orthogonal to the boundary for small time  $t$ , thus the germ of  $g_0$  near the boundary is uniquely determined by  $g$ . We can consider the index problem for  $D_0^+$ , the signature, respectively the Dirac operator associated to  $g_0$ . The boundary condition for  $D_0^+$  is defined by same operator  $A$  as for  $D^+$ , namely the odd signature operator, respectively the Dirac operator on  $(M, h)$ .

**Lemma 12.** *The indices of  $D^+$  and  $D_0^+$  coincide.*

*Proof.* Let  $\{g_s\}_{0 \leq s \leq 1}$  be the segment of metrics linking  $g_0$  to  $g =: g_1$ ,  $g_s = (1 - s)g_0 + sg_1$ . Endow the cylinder  $[0, 1] \times X$  with the metric  $ds^2 + g_s$ . When  $X$  is spin, this cylinder is also spin, moreover when  $\dim(X)$  is even, the spinor bundle on  $[0, 1] \times X$  restricts naturally over every slice  $\{s\} \times X$  to the spinor bundle for  $(X, g_s)$ . In the case of the signature bundle, we can patch together the exterior bundles from the slices  $\{s\} \times X$  to the sub-bundle of the exterior bundle on  $[0, 1] \times X$  consisting of forms annihilated by contraction with the vector field  $\partial_s$ . Thus in both cases, we get a global vector bundle with connection over  $[0, 1] \times X$  extending the family of bundles  $S$  over  $X$  constructed with respect to the metrics  $g_s$ .

Parallel transport on  $[0, 1] \times X$  in this bundle in the direction of  $\partial_s$  identifies the bundle  $S$  over  $(X, g_s)$  with the spinor bundle, respectively the exterior bundle on  $(X, g_0)$ . Since  $\partial_s$  is a geodesic vector field, this parallel transport preserves the horizontal volume form  $dg_s$ , thus it preserves the splitting  $S = S^+ \oplus S^-$  induced by Clifford product with  $dg_s$ . Under these identifications,  $D_s^+$  becomes a first-order elliptic operator of the form (16) near the boundary, smoothly varying in  $s$ . Moreover, we claim that the tangential component at the boundary,  $A_s$ , is constant in  $s$ , being precisely the Dirac operator, respectively the odd signature operator on  $(M, h)$ :

$$D_s^+ = \sigma(s) \cdot [\partial_t + A + tB_1(t, s) + B_0(t, s)].$$

Indeed, the principal symbol of  $A_s$  over  $\{s\} \times M$  is Clifford multiplication in the spinor bundle, respectively in the exterior bundle over  $M$ . But the metrics on  $\{s\} \times M$  are all equal to  $h$  for every  $s$ , hence the family of principal symbols of  $A_s$  is constant. Up to absorbing the zero<sup>th</sup> order part in  $B_0(0, s)$ , we see that  $A$  can be assumed to be constant in  $s$ .

We now have a smooth family of Fredholm operators  $\{D_s^+\}_{s \in [0, 1]}$ ,

$$D_s^+ : \{\phi \in H^1(X, S^+; g_0); \Pi_{[0, \infty)}(\phi|_M) = 0\} \rightarrow L^2(X, S^-; g_0)$$

acting between the same two Hilbert spaces. The main point here is that the spectral projector  $\Pi_{[0, \infty)}$ , defined by  $A$ , is independent of  $s$ . From the homotopy invariance of the Fredholm index we get the equality  $\text{index}(D_0^+) = \text{index}(D^+)$ .  $\square$

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