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Higher transgressions of the Pfaffian

Sergiu Moroianu

Abstract. We define transgressions of arbitrary order, with respect to families of unit-vector fields indexed by a polytope, for the Pfaffian of metric connections for semi-Riemannian metrics on vector bundles. We apply this formula to compute the Euler characteristic of a Riemannian polyhedral manifold in the spirit of Chern's differential-geometric proof of the generalized Gauss-Bonnet formula on closed manifolds and on manifolds-with-boundary. As a consequence, we derive an identity for spherical and hyperbolic polyhedra linking the volumes of faces of even codimension and the measures of outer angles.

1. Introduction

The classical Gauss-Bonnet theorem computes the Euler characteristic of a closed surface (M, g) by integrating on M the Gaussian curvature $\mathfrak{k}_g$. When M has a smooth boundary, one must add a correction term involving the average of the geodesic curvature of the boundary $\partial M \hookrightarrow M$. If M has *corners*, i.e., the boundary itself has isolated singular points, then the exterior angle of these corners must also be taken into account. This general formula reads

$$2\pi\chi(M) = \int_M \mathfrak{k}_g \nu_g - \int_{\partial M} a \cdot l_g + \sum_p \angle^{\mathrm{out}}(p)$$

where $a : \partial M \to \mathbb{R}$ is the geodesic curvature function with respect to the outer normal, ν_g is the volume density, l_g is the length element on ∂M, and $\angle^{\mathrm{out}}(p)$ is the outer angle at a corner p. This outstanding formula has been generalized to arbitrary dimensions by a sequence of authors including H. Hopf, H. Weyl, C. Allendoerfer, A. Weil, and S.S. Chern.

In the present paper we extend the Gauss-Bonnet formula of Allendoerfer and Weil [4] to general compact Riemannian polyhedral manifolds (theorems 6.1 and 6.4).

The outline of the paper is as follows. In the first two sections we review the Pffafian of the curvature using the language of double forms. In Section 4 we define smooth polyhedral manifolds and polyhedral complexes, and study their properties with respect to integration of forms. The category of polyhedral complexes allows one to bundle together the outer cones of faces of Riemannian polyhedral manifolds, the natural locally trivial

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bundles of spherical polytopes where the contributions of the faces are localized. Starting from Chern's construction [11] of a transgression form, we introduce in Section 5 higher transgressions for the Pfaffian form on vector bundles endowed with a nondegenerate bilinear form and a compatible connection. We show that the exterior differential of these transgressions can be computed as a sum of lower-order transgressions. We then apply our abstract transgression theorem in Section 6 to the case of Riemannian polyhedral manifolds. This formula has been proved with entirely different methods by Allendoerfer and Weil [4] for a particular class of polyhedral manifolds that we call *regular*. For regular polyhedral manifolds, the Gauss-Bonnet formula in the even-dimensional case follows by iterating the transgression formula on the boundary strata. In the general case, we use a global polyhedral complex to transfer the transgressions onto the outer cones via the polyhedral Stokes formula. The odd-dimensional case is reduced to the even case by analyzing the Riemannian product with an interval.

In the final section, we particularize our formula to space forms:

Theorem 1.1. *Let M be a d-dimensional compact polyhedral manifold of constant sectional curvature $\mathfrak{k}$, with totally geodesic faces. Then*

$$\frac{\chi(M)}{2} = \sum_{j\geq 0} \sum_{Y\in\mathcal{F}^{(d-2j)}(M)} \mathfrak{k}^j \frac{\mathrm{vol}_{2j}(Y)}{\mathrm{vol}(S^{2j})} \frac{\angle^{\mathrm{out}} Y}{\mathrm{vol}(S^{d-2j-1})} \tag{1.1}$$

where $\mathcal{F}^{(d-2j)}$ is the set of faces of M of dimension $2j$, S^k is the standard unit sphere in $\mathbb{R}^{k+1}$, and $\angle^{\mathrm{out}}Y$ is the measure of the outer solid angle at the face Y.

By convention, for $2j = d$, both the volume of S^{-1} and the exterior angle of the interior face of M are defined to be 1. We deduce from this theorem identities for hyperbolic polyhedra involving the volumes of even-dimensional faces and their outer angles, including an extension to the noncompact case where some - or all - vertices are ideal.

Related works. A strategy of proof of the Gauss-Bonnet formula on polyhedral manifolds similar to the present paper has been proposed in [28]. Wintgen referred to two forthcoming papers of his which were unfortunately never published, due to his untimely demise. Wintgen's normal cycle approach was later generalized to manifolds whose boundary is less regular. There exists a rich bibliography on this subject, cf. [2, 7–10].

The Gauss-Bonnet formula (3.1) continues to hold on complete manifolds with warped product ends with a decay condition on the warping functions [22], and for asymptotically cylindrical metrics [1]. If (M, g) is a smooth compact Riemannian manifold-with-boundary, the Gauss-Bonnet formula contains a correction term along the boundary in terms of the second fundamental form [4, 11]. Extensions of this formula to more general metrics on the interior of a manifold-with-boundary were found by Satake [24] for Riemannian orbifolds, by Albin [1], Dai-Wei [15] and by Cibotaru and the author [14] for manifolds with fibered boundaries, by C. McMullen [18] for cone manifolds, by Anderson [5] for asymptotically hyperbolic Einstein 4-manifolds, and again in [14] for incomplete edge metrics, to cite only a few results in this direction. The proofs typically start from a degeneration process in the Gauss-Bonnet formula for manifolds-with-boundary.

In contrast, the Gauss-Bonnet formula on a Riemannian polyhedral manifold does not seem to follow from such a degeneration. Although it may appear tempting to consider a ϵ-neighborhood of M as a C^1-smoothing of the boundary and then try to compute the limit of the boundary integrand as $\epsilon \to 0$ by interpreting the smoothed boundary as a current like in [13], we were not able to isolate with that approach the contributions of lower-dimensional faces.

Historical note. For submanifolds in $\mathbb{R}^n$, the Gauss-Bonnet formula was stated and proved by Hopf [17] for hypersurfaces, and by Allendoerfer [3] for submanifolds of arbitrary codimension. Allendoerfer and Weil [4] derived their formula on Riemannian polyhedra mainly as a tool for deducing the Gauss-Bonnet on closed Riemannian manifolds without assuming the existence of an isometric embedding in Euclidean space (which we now know to exist by Nash's embedding theorem [21], but was unknown at that time). Their proof is indirect, based on a series of results: a triangulation theorem for polyhedral manifolds, an additivity result for the geometric side of the formula, a proof for simplexes embedded in some Euclidean space using Weyl's tube formula, an embedding theorem for analytic simplexes, predating Nash, due to Cartan, and the Whitney analytic approximation result.

In a series of papers [11], [12], Chern gave an entirely different proof of the Gauss-Bonnet formula, based on his transgression form for the Pfaffian lifted to the sphere bundle. His construction yields simultaneously the necessary correction term on manifolds with boundary, i.e., manifolds with corners of codimension 1.

We extend here Chern's method to transgressions of higher orders (transgressions of transgressions), and use these transgressions to prove the Allendoerfer-Weil formula for a more general class of polyhedral manifolds than that of [4]. In particular, the additivity of the geometric term in the generalized Gauss-Bonnet becomes a corollary of our proof. We close in this way a circle of ideas going back to Gauss and Bonnet, Hopf and Weyl, Allendoerfer and Weil and, last but not least, S.S. Chern. This paper is a tribute to those great mathematicians from the past.

2. Tensor calculus and the Pfaffian

We fix in this section the notation concerning vector bundles with metric connections, and we develop a formalism for multiplying vector-valued forms. Such a formalism was already used by Walter [26] in his generalized Allendoerfer-Weil formula for locally convex subsets in a Riemannian manifold, and also by Albin [1].

2.1. The Pfaffian

Let (V, h) be a $2n$-dimensional real vector space endowed with a nondegenerate symmetric bilinear pairing of signature $(k, 2n-k)$. This means that we can find orthogonal bases $\{e_1, \ldots, e_{2n}\}$ with

$$h(e_1, e_1) = \ldots = h(e_k, e_k) = 1, \qquad h(e_{k+1}, e_{k+1}) = \ldots = h(e_{2n}, e_{2n}) = -1.$$

If moreover V is oriented, the *volume form* defined by h is the unique $2n$-form $\mathrm{vol}_h \in \Lambda^{2n} V^*$ which takes the value 1 on any positively-oriented orthonormal basis. This form defines an isomorphism

$$\mathbb{R} \to \Lambda^{2n} V, \qquad 1 \mapsto \mathrm{vol}_h.$$

The inverse $\mathcal{B}_h : \Lambda^{2n} V \to \mathbb{R}$ of this isomorphism is called the *Berezin integral* with respect to h.

Any skew-symmetric endomorphism $A \in \mathrm{End}^-(V)$ determines a 2-form $\omega_A \in \Lambda^2 V^*$ by

$$\omega_A(u, v) = h(u, Av).$$

The n^{th} power of the 2-form ω_A is a multiple of vol_h. Define the *Pfaffian* of A with respect to h by

$$\mathrm{Pf}(A) = \tfrac{1}{n!}\mathcal{B}_h\left[(\omega_A)^n\right] \in \mathbb{R}.$$

Example 2.1. Let $V = \mathbb{R}^{2n}$ with its euclidean metric and let $\{e_1, \dots, e_{2n}\}$ be the standard basis. The Pfaffian is a homogeneous polynomial of degree n with coefficients ± 1 in the $n(2n-1)$ independent entries of A, containing $(2n-1)!!$ monomials. It is well-known, and easy to prove, that $\mathrm{Pf}(A)^2 = \det(A)$.

For $n = 1$ and $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$, we have $\omega_A = ae^1 \wedge e^2$ and

$$\mathrm{Pf}(A) = a.$$

For $n = 2$ and $A = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & f \\ -b & -d & 0 & g \\ -c & -f & -g & 0 \end{bmatrix}$, we get $\omega_A = ae^1 \wedge e^2 + be^1 \wedge e^3 + ce^1 \wedge e^4 + de^2 \wedge e^3 + fe^2 \wedge e^4 + ge^3 \wedge e^4$, hence

$$\mathrm{Pf}(A) = ag - bf + cd.$$

2.2. Vector-valued forms

Let M be a smooth manifold, and E_1, E_2, E_3 real vector bundles over M. To every linear map $p : E_1 \otimes E_2 \to E_3$ we associate a "product" on spaces of vector-valued forms, i.e., a bilinear map

$$\begin{aligned} P : \Omega^*(M, E_1) \times \Omega^*(M, E_2) &\to \Omega^*(M, E_3), \\ (\alpha_1 \otimes s_1) \times (\alpha_2 \otimes s_2) &\mapsto \alpha_1 \wedge \alpha_2 \otimes p(s_1, s_2). \end{aligned}$$

- A first example of such a product arises for $E_1 = \mathbb{R}$, $E_2 = E_3 = E$ and $p : \mathbb{R} \otimes E \to E$ the canonical isomorphism. We recover the $\Omega^*(M)$-module structure of $\Omega^*(M, E)$.
- When $E_1 = \mathrm{End}(E)$, $E_2 = E_3 = E$, and p is the tautological pairing $\mathrm{End}(E) \times E \to E$, we recover the action of endomorphism-valued forms on E-valued forms.
- A large class of examples arises when $E_1 = E_2 = E_3 = E$ are bundles of $\mathbb{R}$-algebras, and p is the algebra product of E.

- As a particular case of the previous example, let E be the bundle of exterior algebras of a vector bundle V. Set
$$\Omega^{u,v}(M,V) := \Omega^u(M,\Lambda^v V^*).$$
When M and V are clear from the context we will suppress them, using simply the notation $\Omega^{u,v}$. We get a bi-graded algebra structure on the space of *double forms* $\Omega^*(M,\Lambda^*V) = \oplus_{u,v}\Omega^{u,v}$.
- Another particular case: take E to be the endomorphism bundle of some vector bundle V. We get a composition product on the space of *endomorphism valued forms* $\Omega^*(M,\mathrm{End}(V))$.

The last two examples may lead to confusion when V is additionally endowed with a nondegenerate symmetric bilinear pairing h. In that case, there is an identification of the space of h-antisymmetric endomorphisms $\mathrm{End}^-(V)$ with $\Lambda^2(V^*)$, given by

$$\mathrm{End}^-(V) \ni A \mapsto \omega_A \in \Lambda^2(V^*), \qquad \omega_A(u,v) = h(u,Av). \tag{2.1}$$

So there exist two different "product" maps on $\Omega^*(M,\Lambda^2(V^*))$: one is the "exterior product" taking values in $\Omega^*(M,\Lambda^4(V^*))$, the other one is the "composition product" obtained by identifying $\Omega^*(M,\Lambda^2(V^*))$ with $\Omega^*(M,\mathrm{End}^-(V))$ and then using the product of endomorphisms, hence taking values in $\Omega^*(M,\mathrm{End}(V))$.

For simplicity, in the sequel we shall write $\alpha\beta$ for the "product" $P(\alpha,\beta) \in \Omega^*(M,E_3)$.

3. The curvature tensor as a double form

For every vector bundle $E \to M$ with connection ∇ and for every $k \geq 0$ we denote by $\mathrm{d}^\nabla : \Omega^k(M,E) \to \Omega^{k+1}(M,E)$ the exterior differential twisted by ∇ on E-valued forms. Recall that on a tensor product $S = \alpha \otimes s$, where $\alpha \in \Omega^k(M)$ and $s \in \Gamma(E)$ are a locally-defined k-form on M, respectively a section in E, d^∇ is given by
$$\mathrm{d}^\nabla(S) = \mathrm{d}\alpha \otimes s + (-1)^k \alpha\nabla s.$$
The first-order differential operator d^∇ is a derivation on the graded $\Omega^*(M)$-module $\Omega^*(M,E)$, in the sense that for any $\beta \in \Omega^s(M)$ and $S \in \Omega^*(M,E)$, the Leibniz rule holds:
$$\mathrm{d}^\nabla(\beta S) = (\mathrm{d}\beta)S + (-1)^s \beta \mathrm{d}^\nabla S.$$

The composition of two successive operators d^∇ is a 0-th order differential operator, identified with an element $R^\nabla \in \Omega^2(M,\mathrm{End}(E))$. In other words, for all $S \in \Omega^*(M,E)$ we have
$$\mathrm{d}^\nabla(\mathrm{d}^\nabla S) = R^\nabla S$$
where the product is induced from the canonical pairing $\mathrm{End}(E) \times E \to E$. The tensor $R^\nabla \in \Omega^2(M,\mathrm{End}(E))$ is called the *curvature endomorphism* of ∇.

When E has a nondegenerate symmetric bilinear pairing h preserved by ∇, the curvature endomorphism R^∇ will be skew-symmetric. As in (2.1), this endomorphism determines via h a double form
$$\omega_R \in \Omega^{2,2} = \Omega^2(M,\Lambda^2 E^*), \qquad \omega_R(X,Y)(e_1,e_2) = h(e_1, R^\nabla(X,Y)e_2)$$

called the *curvature double form*. When no confusion can arise regarding h, we will continue to denote the form ω_R by R^∇.

3.1. The Pfaffian of the curvature

For any real vector bundle $E \to M$ of even rank $2n$ endowed with a nondegenerate pairing h and a compatible connection ∇, we are ready to define the Pfaffian form $\mathrm{Pf}_h(\nabla)$. When E is oriented, multiplication by the volume form of h defines for every $d \geq 0$ an isomorphism

$$\Omega^d(M) \ni \alpha \mapsto \alpha \otimes \mathrm{vol}_h \in \Omega^{d,2n}.$$

The inverse of this isomorphism is called the Berezin integral $\mathcal{B}_h$ with respect to h. The *Pfaffian form* of ∇ is defined as

$$\mathrm{Pf}_h(R^\nabla) := \tfrac{1}{n!}\mathcal{B}_h(\omega_R^n) \in \Omega^{2n}(M) = \Omega^{2n,0}.$$

When h is implicit from the context, for simplicity we write $\mathrm{Pf}(\nabla)$ for $\mathrm{Pf}_h(R^\nabla)$. Clearly, the Pfaffian form vanishes identically if the dimension of the base M is smaller than the rank of E. For $E = \mathrm{T}M$, even if M is not oriented we can still define the Pfaffian *density* of R, by using the above definition locally with respect to any of the two possible orientations. This is well-defined because $\mathrm{vol}_h \otimes \mathrm{vol}_h$ is invariant under change of orientation.

Lemma 3.1. *If there exists a ∇-parallel section $s \neq 0$ in E, then $\mathrm{Pf}_h(R^\nabla)$ vanishes identically on M.*

Proof. If $\nabla s = 0$, then $R^\nabla s = 0$, thus $s \lrcorner \omega_R = 0$, hence $s \lrcorner (\omega_R^n) = 0$, so $\omega_R^n = 0$. ∎

Theorem 3.2. *The Pfaffian of ∇ is a closed form on M.*

Proof. From the second Bianchi identity, $\mathrm{d}^\nabla \omega_R = 0$, so by the Leibniz rule $\mathrm{d}^\nabla(\omega_R^n) = 0$. Moreover, if $\alpha \in \Omega^{*,2n}$, then one can easily check that $\mathrm{d}\mathcal{B}_h(\alpha) = \mathcal{B}_h(\mathrm{d}^\nabla \alpha)$. Hence

$$\mathrm{d}\mathrm{Pf}_h(\nabla) = \tfrac{1}{n!}\mathrm{d}\mathcal{B}_h(\omega_R^n) = \tfrac{1}{n!}\mathcal{B}_h(\mathrm{d}^\nabla \omega_R^n) = 0.$$ ∎

With these preliminaries, we can now state the classical Gauss-Bonnet formula: Let (M, g) be a compact Riemannian manifold without boundary of dimension $2n$, and $R \in \Omega^{2,2}$ the curvature of a metric connection on $\mathrm{T}M$. Then

$$\int_M \mathrm{Pf}(R) = (2\pi)^n \chi(M). \tag{3.1}$$

The formula is valid even if M is not orientable, in that case $\mathrm{Pf}(R)$ being a density. The reader interested in a short proof of this statement can skip the next section dealing with polyhedral manifolds and complexes thereof, as well as most of Section 5.

Remark. The Pfaffian form on the tangent bundle was introduced by H. Hopf [17] motivated by geometric considerations that can be briefly summarized as follows: The infinitesimal volume of the Gauss map of a compact hypersurface $M \subset \mathbb{R}^{2n+1}$ equals the determinant of the second fundamental form, while the curvature of the hypersurface is, by the Gauss equation, the square of the second fundamental form in the sense of double forms. It follows that the pull-back on M through the Gauss map of the standard volume form from the sphere S^{2n} equals the Pfaffian of the curvature. Hopf computed in this way the degree of the Gauss map intrinsically in terms of the integral of the Pfaffian on M.

4. Polyhedral manifolds

A somewhat informal notion of polyhedral manifold was used in [4]. We give here a rigorous definition together with an extension to a larger category, that of polyhedral complexes.

Polyhedral manifolds extend the notion of manifolds with corners [20]. Since for manifolds with embedded corners the results of the current section become largely obvious, the reader interested only in such manifolds can proceed directly to Section 5.

We refer to [29] for basic properties of polyhedra and polytopes.

4.1. Polyhedral cones

Definition. Let V be a real vector space of dimension n, and $\mathrm{S}V^* = (V^* \setminus \{0\})/\mathbb{R}_+^*$ the dual sphere, consisting of non-zero linear forms on V defined up to a positive constant. Let $A \subset \mathrm{S}V^*$ be a finite set. The *open polyhedral cone* in V defined by A is the cone

$$P_A = \{x \in V; \alpha(x) < 0, (\forall)[\alpha] \in A\}.$$

A non-empty polyhedral cone is always unbounded according to this definition, since it is invariant by dilations with positive factors. The condition $\alpha(x) < 0$ is the same for every representative $\alpha \in [\alpha]$, i.e., it is invariant under rescaling of α by a positive constant.

A *closed polyhedral cone* is the closure in V of an open polyhedral cone. The closure of P_A is clearly

$$\overline{P}_A = \{x \in V; \alpha(x) \leq 0, (\forall)\alpha \in A\}$$

and note that according to this definition, the interior in V of a closed polyhedron is nonempty.

Some of the linear forms defining a polyhedral cone P_A may be redundant, so it is natural to consider *minimal* sets of such defining forms. If a set $A \subset \mathrm{S}V^*$ defines a non-empty open polyhedral cone in V, there exists a unique minimal set $A' \subset A$ defining the same polyhedral cone, i.e., $P_{A'} = P_A$. Two minimal sets $A, A' \subset \mathrm{S}V^*$ define the same polyhedra in V if and only if they are equal.

Let P_A be a nonempty open polyhedral cone defined by a minimal set of linear forms A. For every $\alpha \in A$, the *hyperface* P_A^α is the open polyhedral cone inside the vector space $\ker(\alpha) \subset V$ defined by the relations $\{\beta_{|\ker(\alpha)} < 0; \beta \in A, \beta \neq \alpha\}$. By minimality, this polyhedral cone is non-empty, has dimension $n-1$, and sits inside the closure of P_A. The polyhedral cone P_A has thus as many hypersurfaces as the cardinality of A. Note that the set of defining forms for P_A^α indexed by $A \setminus \{\alpha\}$ may be non-minimal.

Inductively, one defines the faces of depth (or codimension) $l \geq 2$ of a polyhedral cone P as the hyperfaces of the faces of depth $l-1$ of P. A closed polyhedral cone is thus decomposed into the disjoint union of its open faces. We denote

$$\mathcal{F}(P) = \bigcup_{l\geq 1} \mathcal{F}^{(l)}(P)$$

the set of all faces of P of codimension at least 1.

Recall the Minkowski-Weyl theorem [6, 27, 29]: every open polyhedral cone P can be described alternately as the set of linear combinations with positive coefficients of some generating vectors $v_1, \dots v_k$:

$$P = \{c_1 v_1 + \dots + c_k v_k; c_1 > 0, \dots, c_k > 0\}.$$

A minimal set of such generating vectors is unique up to rescaling by positive constants. Conversely, every finite set of vectors spanning V generate by positive linear combinations a polyhedral cone in V. (This polyhedral cone could be the whole of V, corresponding to the empty set of linear forms $A = \emptyset$).

If $P = P_A$ is an open polyhedral cone in V defined by a finite set of forms $A \subset \mathrm{S}V^*$, the *dual polyhedron* P^* is the polyhedral cone inside the dual space V^* defined as the positive linear span of the vectors $\alpha \in A$. For every face $F \subset \overline{P}$, the *conormal space* N^*F is the space of forms in V^* which vanish on F (or equivalently, on its linear span). The intersection of N^*F with P^* is a face of P^*.

4.2. Polyhedra

Definition 4.1. An *open polyhedron* in a vector space V is the set of points $v \in V$ satisfying the inequalities

$$\alpha_j(v) < a_j, \qquad j = 1, \dots, k \tag{4.1}$$

for some $\alpha_1, \dots, \alpha_k \in V^*$ and $a_1, \dots, a_k \in \mathbb{R}$.

Closed polyhedra and their faces are defined as in the case of polyhedral cones. A *polytope* is a compact polyhedron. According to our definition, the interior in V of a nonempty closed polyhedron is always nonempty, and its dimension is $\dim(V)$.

Definition 4.2. The *conormal outer cone* $C^{*\mathrm{out}}(F, P)$ of an open face $F \subset \overline{P}$ in a polyhedron $P \subset V$ is the set of forms $\alpha \in V^*$ such that for every $v \in F$ and $v_0 \in P$, $\alpha(v - v_0) > 0$.

If the polyhedron P is defined by (4.1) and $F \subset \overline{P}$ is the open face determined by

$$F = \{v \in V; \alpha_1(v) = a_1, \dots, \alpha_l(v) = a_l, \alpha_{l+1}(v) < a_{l+1}, \dots, \alpha_k(v) < a_k\}$$

then the conormal outer cone can also be described as the open cone

$$C^{*\mathrm{out}}(F, P) = \{c_1\alpha_1 + \dots + c_l\alpha_l; c_j > 0, (\forall) j = 1, \dots, k\}$$

so in particular $C^{*\mathrm{out}}(F, P)$ is a polyhedral cone in $N^*F \subset V^*$. Let $S^{*\mathrm{out}}(F, P)$ denote the *conormal outer sphere*, defined as $C^{*\mathrm{out}}(F, P)/\mathbb{R}_+^* \subset SV^*$.

If we fix $v \in F$ and $v_0 \in P$, take

$$C_1^{*\mathrm{out}}(F, P) = \{\alpha \in F^{\mathrm{out}} \mid \alpha(v - v_0) = 1\}.$$

Then $C_1^{*\mathrm{out}}(F, P)$ is an open polytope independent of v, and independent of v_0 up to a projective isomorphism. Moreover, for every $v_0 \in P$, $C^{*\mathrm{out}}(F, P)$ is a cone with base $C_1^{*\mathrm{out}}(F, P)$ so the projection defines a canonical homeomorphism from $C_1^{*\mathrm{out}}(F, P)$ to the conormal outer sphere $S^{*\mathrm{out}}(F, P)$. This motivates the definition of a *spherical polytope* in SV^* as the image of a polyhedral cone in V^* by the canonical projection.

A *subcomplex* $C \subset P$ in a closed polyhedron P is the union of some closed faces of P. More generally, a union of closed faces in a product $P \times Q$, where P is a polyhedron and Q is a spherical polytope, is also called a subcomplex.

Definition 4.3. Let $P \subset V$ be a polyhedral cone. The *conormal outer complex*, respectively the *sphere outer complex*, are defined as the disjoint unions

$$C^{*\mathrm{out}}P = \bigcup_{F \in \mathcal{F}(P)} F \times C^{*\mathrm{out}}(F, P) \subset V \times V^*, \qquad S^{*\mathrm{out}}P = \bigcup_{F \in \mathcal{F}(P)} F \times S^{*\mathrm{out}}(F, P).$$

Lemma 4.4. *If P is a polyhedral cone in V then $C^{*\mathrm{out}}P$ and $S^{*\mathrm{out}}P$ are polyhedral subcomplexes in $V \times V^*$, respectively $V \times SV^*$.*

Proof. Clearly $C^{*\mathrm{out}}P$ is a subcomplex in the n-skeleton of the product polyhedron $P \times P^*$. As for $S^{*\mathrm{out}}P$, it is the projection of $C^{*\mathrm{out}}P$ on $P \times SV^*$. ∎

The intersection of $C^{*\mathrm{out}}P$ with the conic $\{(v, \alpha) \in V \times V^*; \alpha(v_0 - v) = -1\}$, defined for some fixed $v_0 \in P$, is the union of the polyhedra $C_1^{*\mathrm{out}}(F, P)$ over all faces F of P. It is canonically identified with $S^{*\mathrm{out}}P$ through the radial projection.

4.3. Polyhedral manifolds and complexes

Let $P \subset V$, $P' \subset V'$ be open polyhedra and $C \subset \overline{P}, C' \subset \overline{P'}$ subcomplexes. A map $f : D \subset C \to C'$ is called *smooth* if for every $x \in D$ there exists an open neighborhood $x \in U \subset V$ and a smooth map $F : U \to V'$ extending $f_{|U \cap D} : U \cap D \to C'$ such that $F(U \cap P) \subset P'$.

It is natural to define a *topological polyhedral manifold* X of dimension n as a separated topological space locally homeomorphic to some closed polyhedron of dimension n, i.e., every point $x \in X$ has a neighborhood homeomorphic to an open set in a closed polyhedron (depending on x) of dimension n. These local homeomorphisms are called *charts*, and a collection of charts covering X forms an *atlas*, which will be required below to have smooth transition maps (without this requirement, a topological polyhedral manifold being just a topological manifold with boundary). Since every polyhedron is locally homeomorphic to a polyhedral cone by an affine transformation, we could have used polyhedral cones as model spaces in this definition.

A *topological polyhedral complex* inside a polyhedral manifold M is a space C so that the pair (M, C) is locally homeomorphic to a subcomplex inside some closed polyhedron. More precisely, for every $x \in C$ there exists a polyhedron P_x, a subcomplex $C_x \subset P_x$, and a homeomorphism, called *chart of complexes*, from a neighborhood $x \in D \subset M$ of x to an open set $U \subset P_x$, such that $D \cap C$ is mapped homeomorphically onto $U \cap C_x$.

Definition. A *polyhedral manifold* (respectively a *polyhedral complex*) is a topological polyhedral manifold (respectively a topological polyhedral complex) endowed with a smooth atlas.

Fix an inner product on a finite-dimensional real vector space V. Then the intersection of a polyhedral cone $P \subset V$ with the unit sphere in V is a polyhedral manifold diffeomorphic (by radial projection) to the polytope

$$P_1 = \{v \in P; \alpha_0(v) = -1\}$$

for some fixed α_0 in the dual polyhedron P^*. This polytope is also diffeomorphic through the natural projection to the spherical polytope $(P \setminus \{0\})/\mathbb{R}_+^*$.

A point of depth $l \geq 0$ in a polyhedral manifold M is a point mapped to a point of depth l through one (and hence any) chart. A *open face of codimension* l of M is a connected component of the set of points of depth l. Such a face is clearly a smooth manifold of dimension $n - l$. However, its closure is in general *not* a polyhedral manifold of the same dimension!

For a face Y inside a polyhedron M, we denote by $\mathcal{F}^{(l)}(Y)$ the set of open faces of M which lie inside $\overline{Y}$ and have codimension $l \geq 0$ in $\overline{Y}$.

Definition 4.5. A polyhedral manifold M is *regular* if the closure of every open face of M is again a polyhedral manifold.

Polyhedral cones and spherical polytopes are examples of regular polyhedral manifolds; the region in the plane bounded by a smooth segment self-intersecting orthogonally in its end-points is an example of non-regular polyhedral surface. Regular polyhedral manifolds are an extension of the notion of manifolds-with-corners with embedded faces. The polyhedral manifolds considered in [4] seem to be regular, although the authors are imprecise on this aspect.

4.4. Outer spheres and the outer cone complex

Let x be a point in a face Y of a polyhedral manifold M. An *interior vector* at x is the tangent vector in 0 to a smooth curve $c : [0, \epsilon) \to M$ with $c(0) = x$. These vectors span the tangent space $\mathrm{T}_x M$.

A Riemannian metric on a polyhedral manifold M is a family of inner products on the tangent spaces to M which, in any chart modeled by a polyhedron in a vector space V, extends to a smooth metric on an open set of V. Riemannian metrics can be constructed on any polyhedral manifold using partitions of unity.

Definition 4.6. Let $y \in M$ be a point in the Riemannian polyhedral manifold M, and let Y be the unique open face of M of which y is an interior point. The *outer cone at* y, denoted $C_y^{\mathrm{out}}(Y, M)$, is the set of those vectors in $\mathrm{T}_x M$ whose inner products with every interior tangent vector at y are non-positive. The *outer sphere* $S_y^{\mathrm{out}}(Y, M)$ is the set of unit vectors in the outer cone at y.

Every vector $V \in C_y^{\mathrm{out}}(Y, M)$ is orthogonal to Y, so $S_y^{\mathrm{out}}(Y, M)$ is a subset of the normal sphere to Y at y (otherwise, the projection of V on Y would have positive inner product with V). Thus $S_y^{\mathrm{out}}(Y, M)$ is a spherical polytope inside $\mathrm{N}_y Y$, the orthogonal complement in $\mathrm{T}_y M$ to $\mathrm{T}_y Y$. Note that when M is a polyhedron inside a metric vector space V, the inner product induces canonical identifications of $\mathrm{N}_y Y$, $C_y^{\mathrm{out}}(Y, M)$ and $S_y^{\mathrm{out}}(Y, M)$ respectively with N^*Y, $C^{*\mathrm{out}}(Y, M)$ and $S^{*\mathrm{out}}(Y, M)$ defined in the previous section.

The conormal space $N_y^* Y$ at a point $y \in Y$ in the interior of a face of a polyhedral manifold M is defined as the set of 1-forms in $\mathrm{T}_y^* M$ annihilating $\mathrm{T}_y Y$. It is straightforward to define the conormal cone $C^{*\mathrm{out}}_y(Y, M)$ and the conormal sphere $S^{*\mathrm{out}}_y(Y, M)$. In the presence of a metric, they are canonically identified with $C_y^{\mathrm{out}}(Y, M)$ and $S_y^{\mathrm{out}}(Y, M)$.

An isomorphism of spherical polytopes is a map induced from a vector space isomorphism of the defining vector spaces. A bundle of spherical polytopes is a locally trivial fibration with fiber type a spherical polytope, and transition maps given by families of isomorphisms of spherical polytopes.

Proposition 4.7. *Let (M, g^M) be a Riemannian polyhedral manifold of dimension n and $Y \subset M$ an open face of codimension $l \geq 1$. Then*

$$S^{\mathrm{out}}Y = \bigsqcup_{y \in Y} S^{\mathrm{out}}_y Y \subset \mathrm{T}M$$

is a bundle of spherical polytopes over Y with fiber type the polytope $Y^* = S^{\mathrm{out}}_{y_0}(Y, M)$ for some fixed $y_0 \in Y$. If M is regular, then $S^{\mathrm{out}}Y \simeq Y \times Y^*$ is globally trivial and the trivialization extends to the closure of Y in M.

Proof. The image of $C^{\mathrm{out}}(Y, M)$ through the isomorphism $\mathrm{T}M \to \mathrm{T}^*M$ induced by g^M is the disjoint union $\bigsqcup_{y\in Y} C^{*\mathrm{out}}_y(Y, M)$, a subset of the normal vector bundle N^*Y over Y.

Choose a chart $\phi : D \to \overline{P}$ with values in a closed polyhedron $\overline{P} \subset \mathbb{R}^n$, and a Riemannian metric g^P on $\mathbb{R}^n$ whose pull-back through ϕ is g^M. Then $D \cap Y$ is mapped into a face F of P of codimension l. Since $\phi_* : \mathrm{T}D \to \mathrm{T}P$ is a vector bundle isomorphism over its image, its dual $\phi^* : \mathrm{T}^*P \to \mathrm{T}^*D$ is also an isomorphism. Since ϕ maps faces of M into faces of P, it follows that ϕ^* descends to an isomorphism between conormal bundles $\phi^* : N^*Y \to N^*F$ preserving the outer conormal cones, and so it also identifies the outer spheres $S^{*\mathrm{out}}_y(Y, M)$ to $S^{*\mathrm{out}}_{\phi(y)}(F, P)$ for every $y \in Y \cap D$.

Let $\{\alpha_1, \dots, \alpha_k\}$ be a minimal system of generating 1-forms for the conormal outer cone $C^{*\mathrm{out}}(F, P)$. Two generating forms $\alpha_{j_0}, \alpha_{j_1}$ are said to be *equivalent* if there exists $s \geq 2$ and indices $j_2, \dots, j_s$ such that the $s + 1$ forms $\alpha_{j_0}, \dots, \alpha_{j_s}$ span a vector space of dimension s, and every s among them are linearly independent. This means that up to rescaling there exists precisely one linear relation $c_{j_0}\alpha_{j_0} + \ldots + c_{j_s}\alpha_{j_s} = 0$ between $\alpha_{j_0}, \dots, \alpha_{j_s}$, and moreover all its coefficients are non-zero. We generate in this way an equivalence relation on the set of generators for the faces of P containing F, i.e., a partition

$$\{1, \dots, k\} = A_1 \sqcup \ldots \sqcup A_q$$

such that N^*F splits as the direct sum of the q linear spans $V_r = \mathrm{span}(\alpha_j; j \in A_r)$.

Consider another chart ϕ' near some $a \in D \cap Y$. By post-composing it with the linear isomorphism $A = \phi_{a*} \circ (\phi'_{a*})^{-1} \in \mathrm{Gl}_n(\mathbb{R})$ with constant coefficients, we construct a chart $A \circ \phi'$ which takes values in the same polyhedron P as ϕ. Since the face Y is connected, we can cover it with domains of charts with values in P, so we can assume that ϕ' takes values in P as well. In this situation, $A = \phi_{a*} \circ (\phi'_{a*})^{-1}$, which *a priori* permutes the directions of the generating forms α_j, must preserve the equivalence relation on generating forms, so in fact it permutes the components V_r of N^*F. The map $\Phi = A \circ \phi' \circ \phi^{-1}$ is the restriction to P of a smooth diffeomorphism of an open set in $\mathbb{R}^n$, it fixes $\phi(a)$, it maps P into itself, and its Jacobian at $\phi(a)$ is the identity map. The hyperfaces of P which contain F are therefore individually preserved by Φ. At the level of conormal bundles, this means that there exist positive rescaling factors $\lambda_j(z) \in \mathbb{R}^*_+$, $z \in F$, such that

$$\Phi^*\alpha_j = \lambda_j(z)\alpha_j. \tag{4.2}$$

Whenever there exists a unique (up to rescaling) nonzero linear relation among a set of equivalent generating forms, by applying Φ to it and using the identities (4.2) it follows that the rescaling factors corresponding to equivalent forms must be equal: $\lambda_{j_0}(z) = \ldots =$

$\lambda_{j_s}(z)$. Hence for every $r = 1, \ldots, q$, the coefficients $\lambda_j(z)$ corresponding to equivalent defining forms are independent of $j \in A_r$, so the map Φ^* preserves the subbundles V_r, $1 \leq r \leq q$ and it acts by dilations on each factor.

The same argument shows that an automorphism of a factor V_r which permutes the directions of the generators $\alpha_j \in V_r$ must be a dilation, and that an automorphism of $\mathrm{N}^*(Y, P)$ permuting the directions of the generators α_j must in fact permute the entire factors V_r, $1 \leq r \leq q$.

It follows that N^*Y is defined by a Čech cocycle consisting of block-diagonal positive dilations Φ composed with isomorphisms A of the blocks with constant coefficients. Using the metric on M, such a cocycle can be retracted to a cocycle consisting of constant transformations permuting the blocks, meaning that we obtain a flat connection on $\mathrm{N}^*(Y, M)$. This connection trivializes locally the outer sphere bundle.

Assume now that M is regular. The closure of every open face is then embedded in M, and hence the unit outer normal vector field to every open hyperface extends continuously to the closure. It follows that every closed face Y is a connected component of an intersection of hyperfaces and we have global generators α_j of the conormal cones along $\overline{Y}$. Then $\mathrm{N}^*(Y, M)$ splits on $\overline{Y}$ as the direct sum of the V_r factors, and by choosing an appropriate atlas on Y which fixes the directions of the generators α_j, we find a Čech cocycle consisting of block-diagonal positive dilations. By using the metric, we can trivialize each factor, so we get a global trivialization of $\mathrm{N}^*(Y, M) \to \overline{Y}$ which in turn induces a trivialization of the outer sphere bundle. ■

A trivialization of the bundle of outer spheres $S^{\text{out}}Y$ is a map V_Y from the model outer sphere $Y^* = S^{\text{out}}_{y_0}(Y, M)$ into the space of unit outer vector fields along Y.

The contribution in the Gauss-Bonnet formula of a face $Y \subset M$ of codimension l will turn out to be given by an integral on the total space of the bundle of outer spheres $S^{\text{out}}Y$, or equivalently the integral on Y of a transgression of order l of the Pfaffian with respect to the family V_Y indexed by Y^* of outer unit vector fields along Y.

Proposition 4.8. *Let (M, g^M) be a Riemannian polyhedral manifold of dimension n. Then the set of outer-pointing unit tangent vectors, denoted M^{out}, is a polyhedral complex of dimension $n - 1$ inside $\mathrm{T}M$. The open top-dimensional faces of M^{out} are the interiors of the outer sphere bundles $S^{\text{out}}Y$, where Y spans all the open faces of M of codimension $l \geq 1$.*

Proof. Suppose first that M is the closure of an open polyhedron in $\mathbb{R}^n$ defined by a minimal set of linear forms $\alpha_1, \ldots, \alpha_p \in A$. If $Y \subset M$ is a face of codimension l, let $A_Y \subset A$ be the set of those α_j's which vanish along Y. After relabeling, we can assume that $A_Y = \{\alpha_1, \ldots, \alpha_k\}$. Proposition 4.7 tells us that $C^{\text{out}}Y$ is isomorphic to $Y \times Y^*$, where the polytope Y^* is the outer sphere at some fixed point $a \in Y$. Moreover, $S^{\text{out}}_y Y$ is spanned as a spherical polytope by the unit vectors $V_1(y), \ldots, V_k(y)$ dual via g^M to $\alpha_1 \ldots, \alpha_k$. It follows that $S^{\text{out}}_y Y$ is diffeomorphic to the conormal outer cone Y^{out} under the diffeomorphism $\mathrm{T}M \to \mathrm{T}^*M$ induced by the metric g^M. Hence $C^{\text{out}}Y$, respectively $S^{\text{out}}Y$ are diffeomorphic to $Y \times Y^{\text{out}}$, respectively to $Y \times Y^{\text{out}}_1$. Thus the claim follows from Lemma 4.4.

In general, M is only locally diffeomorphic to a polyhedron P, thus M^{out} is locally diffeomorphic to a subcomplex in $\mathrm{T}P$, which by definition means that it is a polyhedral complex. ■

4.5. Boundaries of polyhedral complexes and the Stokes formula

It is straightforward to define smooth differential forms, the exterior derivative, and restriction of forms to faces of polyhedral manifolds. We prove below that Stokes formula also continues to hold on polyhedral complexes, once we properly define the boundary of a face.

Let M be a polyhedral complex of dimension $k \geq 1$ and assume that we fix an orientation on all the faces of M of dimension k and $k-1$. Let $F \subset M$ be an open face of dimension $k-1$. We define an integer $\mu_M(F)$, the *multiplicity* of F in M, counting how many times F appears as the oriented boundary of faces of M, as follows: take $x \in F$ and a connected chart in M near x, $\phi : D \to P$, mapping a neighborhood of x into the k-skeleton of a polyhedron P. The chart induces orientations on the k-dimensional faces $Y_1, \ldots, Y_s$ of $P \cap \phi(D)$ whose closure contains the image of x, and also on the unique face $F' \subset P \cap \phi(D)$ of dimension $k-1$ containing $\phi(x)$. For $j = 1, \ldots, s$ let ν_j be a vector field along F' pointing inside Y_j. Let $\mu_x(F', Y_j) \in \{\pm 1\}$ be 1 if the orientations on F' and Y_j are compatible (i.e., $\nu_j, e_1, \ldots, e_{k+1})$ is a negatively oriented frame in $\mathrm{T}_x Y_j$ whenever $(e_1, \ldots, e_{k+1})$ is a positively oriented frame in $\mathrm{T}_x F'$) and -1 otherwise. Define

$$\mu_x(F, M) = \sum_{j=1}^{s} \mu(F', Y_j) \in \mathbb{Z}.$$

This quantity is locally constant on the connected face F, hence it is a constant, denoted $\mu(F, M) \in \mathbb{Z}$. We define the $(k-1)$-boundary of M as the divisor (weighted formal sum of hyperfaces of M):

$$\partial_{k-1}(M) = \sum_{F \in \mathcal{F}^1(M)} \mu(F, M) \cdot F.$$

Note that for a polyhedral manifold N of dimension n with orientable interior, N^{out} is a polyhedral complex of dimension $n-1$ with a natural orientation on its top-dimensional faces, and

$$\partial_{n-2} N^{\text{out}} = 0,$$

in other words the complex N^{out} is a cycle. This cycle appears to be closely related to Wintgen's normal cycle [28].

Lemma 4.9. *Let M be an oriented polyhedral complex of dimension k and $\omega \in \Omega^*(M)$ a compactly supported form. Then*

$$\int_M \mathrm{d}\omega = \int_{\partial_{k-1} M} \omega.$$

Proof. By local considerations involving partitions of unity, it is enough to prove the claim when the form ω is supported in a chart domain. We may therefore assume that M is a subcomplex of dimension k in a polyhedron P, and moreover that ω is supported in a small ball which only intersects those faces passing through its center. On the intersection of this ball with each open k-dimensional face Y we compute the integral of $d\omega$ using the usual Stokes formula, obtaining the integral of ω on the $k-1$-dimensional faces F bounding Y, with a sign depending on the compatibility between the orientations on Y and F. When summing over all Y for a fixed F, we get $\mu(F, M)$ times the integral of ω on F. ■

5. Transgression forms

Let X be an l-dimensional polytope (see Section 4 for the definition), in particular X could be a simplex. Let $E \to M$ be a real vector bundle of rank $2n$ endowed with a pseudo-metric and a compatible connection over a smooth manifold M of arbitrary dimension.

Let V be a family indexed by X of unit-length sections in E over M, i.e., a section $V : X \times M \to \pi_2^* E$ in the pull-back of E to $X \times M$ through the projection $\pi_2 : X \times M \to M$, such that $h(V, V) = 1$. In particular, for every $x \in X$, $V(x, \cdot)$ is a section in E, so V can be viewed as a map $V \in C^\infty(X, \Omega^{0,1}(M, E))$ (remember that E is identified with E^* using h). The connexion ∇ in E acts of such section-valued maps, and $\nabla V \in C^\infty(X, \Omega^{1,1}(M, E))$ is a family of $(1, 1)$-forms indexed by X. Let $d^X V$ be the differential along X of the $\Omega^{0,1}(M, E)$-valued function V, and $(d^X V)^l$ its top-dimensional exterior power. In local coordinates,

$$d^X V := \sum_{j=1}^{l} dx^j \otimes \partial_{x_j} V \in \Omega^1(X, \Omega^{0,1}(M, E)),$$

$$(d^X V)^l = l! dx^1 \wedge \ldots \wedge dx^l \otimes \partial_{x_1} V \wedge \ldots \wedge \partial_{x_l} V \in \Omega^l(X, \Omega^{0,l}(M, E)).$$

For integers l, k, n satisfying $0 \le l \le 2k + 1 \le 2n - 1$, define a universal constant $c(n, k, l)$ by

$$c(n, k, l) = \tfrac{2^k k!}{(n-1-k)!(2k+1-l)!} \in \mathbb{Q}.$$

Definition 5.1. For $l \ge 0$, the $(l + 1)^{\text{th}}$ transgression of the Pfaffian with respect to the family $V : X \to \Omega^{(0,1)}(M)$ is the form

$$\mathcal{T}_X^{(l+1)}(V) = \sum_{l \le 2k+1 \le 2n-1} \tfrac{c(n,k,l)}{l!} \int_X \mathcal{B}[V(d^X V)^l (\nabla V)^{2k+1-l} R^{n-1-k}] \in \Omega^{2n-l-1}(M).$$

The double forms degrees of the objects inside the bracket are: $V \in \Omega^0(X, \Omega^{0,1}(M, E))$, $(d^X V)^l \in \Omega^l(X, \Omega^{0,l}(M, E))$, $R \in \Omega^{2,2}(M, E)$, and $\nabla V \in \Omega^0(X, \Omega^{1,1}(M, E))$, so $\mathcal{B}$ is applied to a volume form on X tensored with a double form in $\Omega^{2n-l-1,2n}(M, E)$, yielding a volume form on X tensored with a form of degree $2n - l - 1$ on M.

For $l = 0$, the first transgression of the Pfaffian of the Levi-Civita connection on the sphere bundle of a Riemannian manifold was introduced by Chern [11].

Functoriality. Like the Pfaffian, the transgression forms are functorial: Let $E \to M$ be a vector bundle with metric and connection, and $\Phi : N \to M$ a smooth map. We equip $\Phi^* E$ with its pull-back metric and connection. For every family V of unit sections in E indexed by X, we consider the family $\Phi^* V$ of unit sections in $\Phi^* E$. In this framework,

$$\mathcal{T}_X^{(l+1)}(\Phi^* V) = \Phi^* \left(\mathcal{T}_X^{(l+1)}(V)\right) \in \Omega^{2n-l-1}(N).$$

We are now ready to prove our main result about transgressions.

Theorem 5.2. *Let $E \to M$ be a vector bundle endowed with a semi-riemannian metric h and a compatible connection ∇, X an oriented polytope of dimension l, and $V : X \times M \to E$ a family of unit-length sections in E indexed by X. Then*

$$\mathrm{d}\mathcal{T}_X^{(l+1)}(V) = \begin{cases} -\mathcal{T}_{\partial X}^{(l)}(V_{|\partial V}) & \text{for } l = \dim X \geq 1, \\ -\mathrm{Pf}(\nabla) & \text{for } X = *, \text{ i.e., } l = \dim(X) = 0, \end{cases}$$

where $\mathcal{T}_{\partial X}^{(l)}(V_{|\partial V}) \in \Omega^{2n-l}(M)$ is the sum of the transgression forms corresponding to the restrictions of V to the oriented hyperfaces of X, i.e., the transgression corresponding to the boundary cycle of X.

Proof. Let $\pi : SE \to M$ be the locally trivial bundle of unit (pseudo-)spheres in E with respect to h. The tangent bundle to SE contains the vertical tangent bundle to the fibers. There is a natural horizontal complement to this vertical bundle, defined by using the connection ∇: the horizontal lift of a path $\gamma : \mathbb{R} \to M$ at a point $v \in S_{\gamma(0)}M$ is the parallel transport of v along γ. Thus ∇ induces a splitting of $\mathrm{T}SE$ as

$$\mathrm{T}SE = \mathrm{T}^{\mathrm{vert}}SE \oplus \pi^*\mathrm{T}M.$$

In the vector bundle $\pi^*E \to SE$ we have the tautological section s of h-length 1, defined by $s_v := v \in E_{\pi(v)} = (\pi^*E)_v$. This connection preserves the pull-back metric h, still denoted by the same symbol.

Lemma 5.3. *Let $\nabla^1 = \pi^*\nabla$ be the pull-back connection in $\pi^*E \to SM$. Then*

$$\nabla^1 s = I_{\mathrm{T}^{\mathrm{vert}}SE} \tag{5.1}$$

in the sense that $\nabla^1_U s = U$ for every $U \in \mathrm{T}_v^{\mathrm{vert}}SE = \mathrm{T}_v(S_v M) \subset E_v$.

Proof. Essentially by definition, the canonical section s is parallel in horizontal directions with respect to ∇^1, so $\nabla^1 s$ is a vertical double form. Also by definition, the pull-back connection is trivial in vertical directions, so we compute $\nabla^1 s = ds$ on each vertical pseudo-sphere $S_v E$, where s becomes a map from $S_v E$ to the fixed vector space E_v. ■

The idea of computing the Pfaffian of ∇^1 (going back to Chern [11]) is to modify ∇^1 on SE so that s becomes parallel. For this, define a vertical $(1,2)$-double form $\alpha \in \Omega^1(SE, \Lambda^2\pi^*E)$ by

$$\alpha = s \cdot \nabla^1 s, \qquad \alpha(U) = s \wedge U \in \Lambda^2\pi^*E$$

for every vertical vector $U \in \mathrm{T}^{\mathrm{vert}}SE$, while $\alpha(H) = 0$ for horizontal $H \in \mathrm{T}^{\mathrm{hor}}SE$. Denote by A the skew-symmetric endomorphism-valued 1-form associated to α via h as in (2.1):

$$\alpha(U)(V,W) = \langle V, A(U)W\rangle_h, \qquad (\forall)U \in \mathrm{T}SE, (\forall)V, W \in \pi^*E, \tag{5.2}$$

so $\alpha = \omega_A$ using the notation from (2.1). (We identify E with E^* via the musical isomorphism in terms of h, thus $\Lambda^2\pi^*E \simeq \Lambda^2\pi^*E^*$). Then we can rewrite (5.1) as

$$\nabla^1 s = -As.$$

For $t \in \mathbb{R}$, set
$$\nabla^t := \nabla^1 + (1-t)A.$$
Since A is skew-symmetric and ∇^1 preserves h, so does ∇^t for all t. We compute $\nabla^t s = -tAs = tI_{\mathrm{T^{vert}}SE}$, hence $\nabla^0 s = 0$ and thus, by Lemma 3.1, $\mathrm{Pf}_h(\nabla^0) = 0$. We shall recover the Paffian of ∇^1 as the integral from 0 to 1 of the t-derivative of $\mathrm{Pf}(\nabla^t)$.

Consider the connection D on the pull-back bundle $\pi^*E \to \mathbb{R} \times SE$ defined by

$$D := \mathrm{d}t \otimes \partial_t(\cdot) + \nabla^t = \mathrm{d}t \otimes \partial_t(\cdot) + \nabla^1 + (t-1)s \cdot \nabla^1 s. \tag{5.3}$$

This connection also preserves the (pull-back of the) metric h.

Let $V : X \times M \to SE$ be a family of unit sections in E (i.e., for every $x \in X, p \in M$, $V(x,p) \in E_p$ is a unit-length vector). Consider the smooth map
$$\Phi : \mathbb{R} \times X \times M \to \mathbb{R} \times SE, \qquad \Phi(t,x,p) = (t, V(x,p))$$
and let Φ^*D be the pull-back connection in the bundle $\Phi^*\pi^*E = \pi_3^*E$ (where $\pi_3 : \mathbb{R} \times X \times M \to M$ is the projection on the third factor).

By the naturality of curvature and of the Pfaffian, we have
$$\Phi^*\mathrm{Pf}_h(D) = \mathrm{Pf}(\Phi^*D) \in \Omega^{2n}(X \times \mathbb{R} \times M).$$
Integrating this Pfaffian in the $X \times \mathbb{R}$ variables, we obtain a form

$$T_X^\Phi := \int_{[0,1]\times X} \mathrm{Pf}(\Phi^*D) \in \Omega^{2n-l-1}(M). \tag{5.4}$$

We compute
$$\mathrm{d}T_X^\Phi = \int_{[0,1]\times X} \mathrm{d}^M \mathrm{Pf}(\Phi^*D) \in \Omega^{2n-l}(M).$$
The Pfaffian form is closed on $\mathbb{R} \times X \times M$, so Stokes formula on the polyhedral manifold $[0,1] \times X \times M$ implies

$$\begin{aligned}
\mathrm{d}T_X^\Phi &= \int_{[0,1]\times X} -\mathrm{d}t \wedge \partial_t \mathrm{Pf}(\Phi^*D) - \mathrm{d}^X \mathrm{Pf}(\Phi^*D) \\
&= \int_{\{0\}\times X} \mathrm{Pf}(\Phi^*D) - \int_{\{1\}\times X} \mathrm{Pf}(\Phi^*D) - \int_{[0,1]\times \partial X} \mathrm{Pf}(\Phi^*D)
\end{aligned} \tag{5.5}$$

where ∂X is the oriented sum of hyperfaces of X.

Notice that $D_{|\{0\}\times SE} = \nabla^0$. We have seen above that $\mathrm{Pf}(\nabla^0) = 0$ because there exists a non-zero parallel section for ∇^0 on SE, so by naturality of the Pfaffian, $\mathrm{Pf}(\Phi^*D) = 0$ on $\{0\} \times X \times M$.

Similarly, $D_{|\{1\}\times SE} = (0,V)^*\pi^*\nabla = \pi_3^*\nabla$, so $\mathrm{Pf}(\Phi^*D) = \pi_3^*\mathrm{Pf}(\nabla)$ on $\{1\} \times X \times M$. This pull-back form does not contain any $\mathrm{d}x^j$ (where $x_1, \ldots, x_l$ are the euclidean variables on X). Hence for $l > 0$ the integral on $\{1\} \times X$ of the second term also vanishes, while for $l = 0$ (i.e., when X is a point) it reduces to $\mathrm{Pf}(\nabla)$.

The third term from (5.5) is the sum of the forms T_F^Φ corresponding to the oriented hyperfaces F of X, which we denote $T_{\partial X}^\Phi$. Thus (5.5) reduces to

$$\mathrm{d}T_X^\Phi = \begin{cases} -T_{\partial X}^\Phi & \text{for } l \geq 1, \\ -\mathrm{Pf}(\nabla) & \text{for } l = 0. \end{cases}$$

We prove below that the form T_X^Φ equals the transgression $\mathcal{T}_X^{(l+1)}(V)$ from Definition 5.1. Since X was arbitrary, the same identity holds for all faces of X, hence we also have $T_{\partial X}^\Phi = \mathcal{T}_{\partial X}^{(l)}(V_{|\partial V})$, thereby ending the proof.

In order to show that $T_X^\Phi = \mathcal{T}_X^{(l+1)}(V)$ we must compute explicitly the Pfaffian of $\Phi^* D$. This computation is made possible by the commutative nature of the Pfaffian polynomial; note that a similar computation would be unlikely for other characteristic polynomials. From (5.3), it is straightforward to compute the curvature of D on $\mathbb{R} \times SE$:

$$R^D = \pi^* R^\nabla - (t-1)\pi^* R^\nabla s \cdot s + \tfrac{1-t^2}{2} \nabla^1 s \cdot \nabla^1 s + \mathrm{d}t \otimes s \cdot \nabla^1 s.$$

The $\cdot$ products above are in the sense of double forms: $s \in \Omega^{0,1}$, $\pi^* R^\nabla \in \Omega^{2,2}$, $\pi^* R^\nabla s \in \Omega^{2,1}$ (in this term, $\pi^* R^\nabla$ is considered as a $\mathrm{End}(E)$-valued 2-form, which is then applied to s) and $\nabla^1 s \in \Omega^{1,1}$. Notice that the pull-back of $\pi^* E$ through Φ from $\mathbb{R} \times SE$ to $\mathbb{R} \times X \times M$ is just $\pi_3^* E$, where π_3 is the projection on the factor M. Using the somewhat imprecise but suggestive notation ∇ for the pull-back connection $\pi_3^* \nabla$ in $\pi_3^* E$ over $\mathbb{R} \times X \times M$, we get for the curvature of $\Phi^* D$ by naturality:

$$R^{\Phi^* D} = \Phi^* R^D = R^\nabla - (t-1) R^\nabla V \wedge V + \tfrac{1-t^2}{2} (\mathrm{d}^X V + \nabla V)^2 + \mathrm{d}t \otimes V \cdot (\mathrm{d}^X V + \nabla V).$$

We proceed to analyze the n^{th} power of this double form inside the space of double forms $\Omega^{2n,2n}(\mathbb{R} \times X \times M, \pi_3^* E)$. Since double forms of even bi-order (i.e., the sum of orders is even) form a commutative subalgebra, we treat this power as a homogeneous polynomial of degree n in the four components of degree $(2,2)$ in $\Phi^* R^D$. We are only interested in those monomials which are multiples of the volume form of $\mathbb{R} \times X$, and clearly those terms must contain precisely once the form $\mathrm{d}t$. Thus, the term $\mathrm{d}t \otimes V \cdot (\mathrm{d}^X V + \nabla V)$ appears precisely once, and so the term $(t-1) R^\nabla V \wedge V$ does not contribute at all (since it contains V, which already appeared in the former term, while for a monomial of top fiber degree $2n$ to be nonzero, the section V may occur at most once.) In conclusion, those monomials from $(\Phi^* R^D)^n$ containing the volume form of $\mathbb{R} \times X$ are contained in

$$n \mathrm{d}t \otimes V \cdot (\mathrm{d}^X V + \nabla V) \left[R^\nabla + \tfrac{1-t^2}{2} (\mathrm{d}^X V + \nabla V)^2 \right]^{n-1}.$$

Using the binomial formula, we write the above as

$$n \mathrm{d}t \otimes V \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(1-t^2)^k}{2^k} (\mathrm{d}^X V + \nabla V)^{2k+1} \left(R^\nabla \right)^{n-k-1}.$$

Apply again the binomial formula to the term $(\mathrm{d}^X V + \nabla V)^{2k+1}$, and retain only the term of top degree l in the X variables, since we eventually need to integrate over X. Hence T_X

from (5.4) is computed as

$$\sum_{k=\lceil\frac{l+1}{2}\rceil}^{n-1}\int_0^1(1-t^2)^k\mathrm{d}t\cdot n\binom{n-1}{k}2^{-k}\binom{2k+1}{l}\frac{1}{n!}\int_X \mathcal{B}\Big[V(\mathrm{d}^X V)^l(\nabla V)^{2k+1-l}(R^\nabla)^{n-k-1}\Big].$$

This gives precisely the transgression $\mathcal{T}_X^{(l+1)}(V)$ from Definition 5.1, since

$$\int_0^1(1-t^2)^k\mathrm{d}t=\frac{2k(2k-2)\dots 2}{(2k+1)(2k-1)\dots 3\cdot 1}=\frac{(2k)!!}{(2k+1)!!}.$$ ■

6. The Gauss-Bonnet formula on polyhedral manifolds

6.1. The Allendoerfer-Weil formula in even dimensions

Let (M^{2n},g) be a compact Riemannian polyhedral manifold of even dimension $2n$ endowed with a Riemannian metric g. Let $Y\subset M$ be a face of codimension $l\geq 0$, and $\mathrm{N}Y\subset \mathrm{T}M_{|Y}$ the normal bundle of Y inside M with respect to g. The second fundamental form of this inclusion is the symmetric bilinear map

$$A:\mathrm{T}Y\times\mathrm{T}Y\to\mathrm{N}Y, \qquad\qquad A(U,W)=\nabla^M_U W-\nabla^Y_U W,$$

that we interpret as a $\mathrm{N}Y$-valued double form of degree $(1,1)$ on Y by ignoring its symmetry. We construct from A its dual, a smooth section $A^*\in C^\infty(\mathrm{N}Y,\Lambda^{1,1}(Y))$ on the total space of $\mathrm{N}Y$ with values in the pull-back from Y of the bundle of double forms: for $V\in\mathrm{N}_yY$ and $U,W\in\mathrm{T}_yY$,

$$A^*(V)(U,W)=\langle V,A(U,W)\rangle_g \tag{6.1}$$

For any $B,C\in\Omega^1(Y,\mathrm{T}^*Y\otimes\mathrm{N}Y)$ pure tensors of the form

$$B=b_1\otimes b_2\otimes v_1, \qquad\qquad C=c_1\otimes c_2\otimes v_2$$

define the partial contraction with g on the $\mathrm{N}Y$ factor by

$$g(B,C)=g(v_1,v_2)(b_1\wedge c_1)\otimes(b_2\wedge c_2)\in\Lambda^{2,2}(Y). \tag{6.2}$$

This definition allows us to define by linearity $g(A,A)\in\Omega^{2,2}(Y)$.

We are now ready to state and prove the extension of the Allendoerfer-Weil formula [4] for the Euler characteristic of a compact Riemannian polyhedral manifold.

Theorem 6.1. *Let M^{2n} be a compact Riemannian polyhedral manifold. Then*

$$(2\pi)^n\chi(M)-\int_M\mathrm{Pf}(R)=\sum_{l=1}^{2n}\sum_{k=\lceil\frac{l}{2}\rceil}^{n}\frac{(-1)^l2^{k-1}(k-1)!}{(n-k)!(2k-l)!}$$
$$\sum_{Y\in\mathcal{F}^{(l)}(M)}\int_{S^{\mathrm{out}}Y}\mathcal{B}_Y\Big[(R^Y-\tfrac12 g(A,A))^{n-k}(A^*)^{2k-l}\Big]|\mathrm{d}g|.$$

Here $|\mathrm{d}g|$ is the family of spherical densities induced by g on the fibers of the normal sphere bundle $SY \to Y$, while the Berezin integral $\mathcal{B}_Y$ produces a section on the total space of SY in the pull-back from the base of the bundle of volume forms on Y. The above integral can thus be considered either as an integral on the total space of $S^{\mathrm{out}}Y$, or (using Fubini's theorem) as an iterated integral, first along the fibers of $S^{\mathrm{out}}Y \to Y$ and then on Y. The ceiling symbol $\lceil \frac{l}{2} \rceil$ denotes the smallest integer greater than or equal to $l/2$. We use the convention $0! = 1$. Also, the 0^{th} power of a tensor like A^* (appearing for $2k = l$) or $R^Y - \frac{1}{2}g(A, A)$ (for $k = n$) is understood to be always 1, regardless of the possible vanishing of the tensor in question.

Proof. We first give the argument under the assumption that the polyhedral manifold M is regular (Definition 4.5), thus recovering the main result of [4]. The general case requires some additional combinatorial properties of the outer cone complex and will be treated in Section 6.2.

We apply successively the transgression formula from Theorem 5.2 to the vector bundle $\mathrm{T}M$ restricted to the various faces of M in increasing order of codimension.

An outer vector field with nondegenerate zeros. Starting from the lowest dimensional faces of M, we construct a smooth vector field U along the boundary faces of M such that for every boundary point x, $-U_x$ is an interior vector, i.e., it has an integral curve lying in the interior of M. This is possible since the cones of interior vectors are convex. We extend this vector field smoothly to the interior of M, and then perturb it to a vector field $U \in \Omega^0(M, \mathrm{T}M)$ transverse to the zero section $M \subset \mathrm{T}M$. If the perturbation is small enough in C^0 norm, the vector field $-U$ will still point in the interior at every boundary point.

Define a unit vector field $V_0 := |U|^{-1}U$ on the complement on the (isolated) zero-set $Z(U)$ of U in M. It is a section of the sphere bundle $SM \to M$ over the complement of $Z(U)$.

Blow-up of the singular set of V_0. Let $\tilde{M}$ be the closure of $V_0(M \setminus Z(U))$ in the polyhedral manifold SM.

Remark. When M is a manifold with corners, the compact polyhedral manifold $\tilde{M}$ is diffeomorphic to $[M; Z(U)]$, the total space of the blow-up of $Z(U)$ inside M. We review below the notion of real blow-up of manifolds-with-corners, and we refer to [20] for more details.

The boundary of the compactification $\tilde{M}$ in SM is obtained by gluing the tangent sphere S_pM near each annulation point $p \in Z(U)$. More precisely, besides the diffeomorphic image through V_0 of the boundary hyperfaces of M, $\partial\tilde{M}$ contains also the "inner boundary", i.e., the singular divisor obtained by blowing-up the annulation points of U. Near a non-degenerate annulation point $p \in Z(U)$, there exist local coordinates in which the vector field U takes the form

$$U(x) = x_1\partial_{x_1} + \ldots + x_r\partial_{x_r} - x_{r+1}\partial_{x_{r+1}} - \ldots x_{2n}\partial_{x_{2n}}.$$

where the integer r is the index of U at p. The new hypersurface introduced by blowing up p is just the compact manifold S_pM, with orientation $(-1)^{r+1}$ times the standard orientation induced from T_pM. We thus separate the boundary of $\tilde{M}$ into the union of the inner boundary spheres, and the diffeomorphic image through V_0 of the boundary of M:

$$\partial\tilde{M} = \Big(\bigsqcup_{p\in Z(U)} S_pM\Big) \sqcup V_0(\partial M).$$

In order to compute the integral of the Pfaffian on M, we will apply Theorem 5.2 to the pull-back bundle $\pi^*\mathrm{T}M \to \tilde{M}$ over the compact polyhedral manifold $\tilde{M}$, endowed with the pull-back connection $\pi^*\nabla$. This clever construction (due to Chern [11]) is necessary because $M \setminus Z(U)$ is not compact, so the Stokes formula would need to take into account the contribution of the singularities $Z(U)$ in the transgression forms. The rôle of the blow-up space $\tilde{M}$ is precisely to "resolve" this singularity formally.

Since $Z(U)$ is a finite set, it has measure 0. By naturality, the integral of the Pfaffian on M can be computed by pull-back on $\tilde{M}$:

$$\int_M \mathrm{Pf}(R) = \int_{V_0(M\setminus Z(U))} \pi^*\mathrm{Pf}(R) = \int_{\tilde{M}} \mathrm{Pf}(R^{\pi^*\nabla}). \tag{6.3}$$

Let $\mathcal{T}^{(1)}(V_0) \in \Omega^{2n-1}(M \setminus Z(U))$ be the first-order transgression on $M \setminus Z(U)$ from Definition 5.1 corresponding to the unit vector field V_0 interpreted as a 0-dimensional simplex of unit vector fields. Similarly, let $\mathcal{T}^{(1)}(s) \in \Omega^{2n-1}(SM)$ be the first-order transgression from Definition 5.1 corresponding to the canonical unit section s in $\pi^*\mathrm{T}M$ over $\tilde{M} \subset SM$, interpreted as a 0-dimensional simplex of unit sections in $\pi^*\mathrm{T}M$. By the naturality of transgression forms, on the complement of the zero set $Z(U)$ we have $\pi^*\mathcal{T}^{(1)}(V_0) = \mathcal{T}^{(1)}(s)$. Using the Stokes formula, if we denote by $r(p)$ the index of U at a zero $p \in Z(U)$,

$$\begin{aligned}\int_{\tilde{M}} \mathrm{Pf}(R^{\pi^*\nabla}) &= -\int_{\partial\tilde{M}} \mathcal{T}^{(1)}(s)\\ &= -\int_{\partial M} \mathcal{T}^{(1)}(V_0) + \sum_{p\in Z(U)} (-1)^{r(p)} \int_{S_pM} \mathcal{T}^{(1)}(s).\end{aligned}$$

Lemma 6.2 (Chern [11]). *At a annulation point $p \in Z(U)$, the integral on the sphere S_pM of $\mathcal{T}^{(1)}(s)$ equals $(2\pi)^n$.*

Proof. Apply Definition 5.1 of the transgression in dimension $l = 0$ for the canonical unit vector field s over the interior of polyhedral manifold $\tilde{M} \subset SM$. Here the parameter space X is just a point, hence the terms containing $\mathrm{d}^X V$ vanish. The curvature R vanishes on the vertical sphere S_pM since it is the pull-back of the curvature from the base, so the terms with $k < n-1$ also vanish. It follows that

$$\mathcal{T}^{(1)}(s)_{|S_pM} = \tfrac{2^{n-1}(n-1)!}{(2n-1)!}\mathcal{B}\Big[s\big(\pi^*\nabla(s)\big)^{2n-1}\Big]$$

where $\pi^*\nabla(s)$ is given by (5.1), $\pi^*\nabla = \nabla^1$ and $\nabla^1 s_{|\mathrm{T}^{\mathrm{vert}}SM} = I_{|\mathrm{T}^{\mathrm{vert}}SM}$. The lemma follows by noting that the volume of S^{2n-1} equals $\frac{2\pi^n}{(n-1)!}$. ∎

The case where M is a compact manifold with (possibly empty) boundary. It follows from the above lemma and the Poincaré-Hopf theorem that the inner boundary contributions add up, like in the boundary-less case, to $(2\pi)^n\chi(M)$. We thus rewrite (6.3) as

$$(2\pi)^n\chi(M) = \int_M \mathrm{Pf}(R) + \int_{\partial M} \mathcal{T}^{(1)}(V_0). \tag{6.4}$$

This identity finishes the proof of the Gauss-Bonnet theorem for closed manifolds.

If M is compact with boundary, we can choose V_0 to be the unit outer normal to ∂M. The correction term $\mathcal{T}^{(1)}(\nu_{\partial M})$ is computed in that case as in the final part of the present proof.

The general case. When M has faces of codimension ≥ 2, in Eq. (6.3) the contribution $\int_Y \mathcal{T}^{(1)}(V_0)$ of a boundary hyperface Y depends on the choice of the vector field V_0, which cannot be chosen to be the unit normal to Y simultaneously for all hyperfaces Y. In order to write this contribution in terms of the outer unit normal vector field ν_Y, we use the higher transgressions with respect to certain families of unit vector fields $V_Y, V_{0,Y}$ defined for every face Y of codimension $l \geq 1$. First, the family $V_Y : Y^* \to C^\infty(Y, S^{\text{out}}Y)$ is indexed by the spherical polytope $Y^* \subset \mathbb{R}^l$ of dimension $l-1$, and corresponds to the trivialization of $S^{\text{out}}Y$ constructed in Proposition 4.7. As for $V_{0,Y}$, is it the cone of vector fields over V_Y with apex V_0. More precisely, for every spherical polytope Z in $\mathbb{R}^l$, define the spherical cone CZ over Z as the l-dimensional spherical polytope inside $\mathbb{R}^l \times \mathbb{R}$ with apex e_{l+1}:

$$CZ = \{\cos(\alpha)v + \sin(\alpha)e_{l+1} | v \in Z \subset \mathbb{R}, \alpha \in [0, \pi/2]\}. \tag{6.5}$$

Then $V_{0,Y}$ is the map from CY^* to $C^\infty(Y, \mathrm{T}M)$ defined by

$$\cos(\alpha)v + \sin(\alpha)e_{l+1} \mapsto \frac{\cos(\alpha)V_Y(v) + \sin(\alpha)V_0}{\|\cos(\alpha)V_Y(v) + \sin(\alpha)V_0\|}.$$

The non-vanishing of the denominator is a consequence of the fact that $-V_0$ points in the interior of M, while $V_Y(v)$ lives in the outer sphere, and these two sets are disjoint.

By Theorem 5.2,

$$\int_Y \mathcal{T}^{(1)}(V_0) = \int_Y \mathcal{T}^{(1)}(V_Y) - \int_{\partial Y} \mathcal{T}^{(2)}_{CY^*}(V_{0,Y}). \tag{6.6}$$

This identity is meaningful because the first term from the right-hand side does not depend on the choice of V_0, while the second is now localized to faces of codimension 2. The induction procedure is powered by the next result:

Lemma 6.3. *Let $Y = F_1 \cap \ldots \cap F_l$ be a (possibly disconnected) face of M, where $F_1, \ldots, F_l$ are hyperfaces of M. Let $Z_j := \cap_{i \neq j} F_i$. Then*

$$\sum_{j=1}^{l} (-1)^j \int_Y \mathcal{T}^{(l)}(V_{0,Z_j}) = \int_Y \mathcal{T}^{(l)}_{Y^*}(V_Y) - \int_{\partial Y} \mathcal{T}^{(l+1)}_{CY^*}(V_{0,Y}).$$

Proof. Direct application of Stokes formula and Theorem 5.2. ∎

Again by induction, for all $d \geq 0$ we have

$$\int_M \mathrm{Pf}(R) = (2\pi)^n \chi(M) + \sum_{l=1}^{d} \sum_{Y \in \mathcal{F}^{(l)}(M)} \int_Y \mathcal{T}^{(l)}_{Y^*}(V_Y)$$
$$+ \sum_{Y \in \mathcal{F}^{(d)}(M)} \int_{\partial Y} \mathcal{T}^{(d+1)}_{CY^*}(V_{0,Y}).$$

The initial step is Eq. (6.6). Specializing to the maximal codimension $d = 2n + 1$, we have completely eliminated the non-canonical vector field V_0 from the formula!

$$\int_M \mathrm{Pf}(R) = (2\pi)^n \chi(M) + \sum_{l=1}^{2n} \sum_{Y\in\mathcal{F}^{(l)}(M)} \int_Y \mathcal{T}^{(l)}_{Y^*}(V_Y).$$

It remains to identify the contribution of each face in terms of intrinsic and extrinsic geometry of the faces (curvature and second fundamental form). Fix a face Y of codimension $l \geq 1$ in M. For simplicity, we write $V = V_Y$ for family indexed by Y^* of unit outer vector fields in $\mathrm{T}M$ along Y defined by the trivialization $V_Y : Y^* \times Y \to S^{\mathrm{out}}Y$ of the outer sphere bundle of Y. The transgression $\mathcal{T}^{(l)}_{Y^*}(V_Y)$ was defined by

$$\mathcal{T}^{(l)}_{Y^*}(V_Y) = \sum_{k=\lceil \frac{l}{2}\rceil-1}^{n-1} \frac{c(n,k,l-1)}{(l-1)!} \int_{Y^*} \mathcal{B}\Big[V(\mathrm{d}^{Y^*}V)^{l-1}(\nabla V)^{2k+2-l}R^{n-1-k}\Big] \in \Omega^{2n-l}(Y).$$

Here $\mathcal{B}$ is the Berezin integral on the pull-back of the bundle $\mathrm{T}M$ to $Y^* \times Y \simeq S^{\mathrm{out}}Y$. The monomial $\mathrm{d}^{Y^*}V$ is a form on V^* tensored with a vector tangent to the sphere $S^{\mathrm{out}}_x Y$. For $x \in Y$, if we denote by $\nu_{S^{\mathrm{out}}Y}$ the Riemannian volume form on the sphere $S^{\mathrm{out}}_x Y$, a short computation shows that $(\mathrm{d}^{Y^*}V)^{l-1}$ can be expressed as the pull-back on Y^* through the map V of the tensor square of the volume form of outer spheres:

$$(\mathrm{d}^{Y^*}V)^{l-1} = (l-1)!V^*(\nu_{S^{\mathrm{out}}_x Y}) \otimes \nu_{S^{\mathrm{out}}_x Y}.$$

Thus the second component of the double form $V(\mathrm{d}^{Y^*}V)^{l-1}$ is a multiple of the volume form of the normal bundle to Y. It follows that only those terms from ∇V and R whose second component is a horizontal form (i.e., they vanish whenever they are contracted with a vector tangent to the fibers of $S^{\mathrm{out}}Y \to Y$) may have a nonzero contribution to $\mathcal{T}^{(l)}_{Y^*}(V_Y)$. These terms are $A^*(V)$, the second fundamental form (6.1) of Y in M interpreted as a $(1,1)$ form-valued function on $\mathrm{N}Y$, and also $R_{|Y}$, the components of the curvature form of M along Y. Recall that by the Gauss equation

$$R_{|Y} = R^Y - \tfrac{1}{2}g(A, A)$$

where the contraction $g(A, A)$ was defined in (6.2). For $x \in Y$ we obtain by changing variables in the integral from the polytope Y^* to the outer sphere $S^{\mathrm{out}}_x Y$ using the diffeomorphisms $V : Y^* \to S^{\mathrm{out}}_x Y$:

$$\begin{aligned}\mathcal{T}^{(l)}_{Y^*}(V_Y)(x) = \sum_{k=\lceil \frac{l}{2}\rceil-1}^{n-1} c(n,k,l-1) \int_{S^{\mathrm{out}}_x Y} \nu_{S^{\mathrm{out}}_x Y} \otimes 1\cdot \\ \mathcal{B}\Big[1 \otimes \nu_{\mathrm{N}Y} \cdot (-A^*)^{2k+2-l}(R^Y - g(A,A)/2)^{n-1-k}\Big].\end{aligned}$$

The proof is finished by noting that $\mathcal{B}(1 \otimes \nu_{\mathrm{N}Y} \cdot \alpha) = \mathcal{B}_Y(\alpha)$ for every double form α whose second component is tangential to Y. ∎

6.2. Passing from regular to general polyhedral manifolds

If M is a not regular polyhedral manifold, the above proof breaks down because the outer cone bundles are not globally trivial. Thus we need a new global argument before applying the local computations from the previous sections. Let I denote the unit interval $[0,1]$.

In the pull-back of $\mathrm{T}M$ over the polyhedral manifold $SM\times I_x\times I_t$ we consider the pull-back $\mathcal{D}$ of the connection D from (5.3) under the projection off the factor $[0,1]_x$ onto $SM\times I_t$:

$$\mathcal{D} = \pi^*\nabla + (1-t)A$$

where π is the projection $SM\times I^2 \to M$, and A is the endomorphism-valued 1-form defined in (5.2) with respect to the tautological section s in $\pi^*\mathrm{T}M$,

$$A(W) = s\wedge \pi_* W \in \mathrm{End}^-(\pi^*\mathrm{T}M).$$

Here t is a deformation parameter as before, while $x\in[0,\pi/2]$ will be the variable of a conical deformation of the polyhedral complex M^{out} that we now introduce. Recall that we have fixed a vector field U on M with isolated nondegenerate zeros and such that $-U$ is inward-pointing along ∂M, and we constructed $V_0 = U/\|U\|$ on the complement of $Z(U)$. In particular, for every $p\in Y\in\mathcal{F}(M)$, $-V_0(p)$ does not intersect the convex spherical polytope $S^{\mathrm{out}}_p Y$, where $\mathcal{F}(M)$ is the set of faces of M of codimension at least 1.

For every face $Y\in\mathcal{F}(M)$ define a locally trivial bundle of spherical polyhedra with fiber type the spherical cone over $S^{\mathrm{out}}_{p_0}Y$:

$$\mathrm{Con}_{V_0}(S^{\mathrm{out}}Y) = \Big\{\Big(\frac{\cos x\cdot v_p + \sin x\cdot V_0(p)}{\|\cos x\cdot v_p + \sin x\cdot V_0(p)\|}, x\Big); p\in Y, v_p\in S^{\mathrm{out}}Y, x\in[0,\pi/2]\Big\}.$$

From Proposition 4.8, it follows that the set

$$\mathrm{Con}_{V_0}(M^{\mathrm{out}}) = \bigcup_{Y\in\mathcal{F}(M)} \mathrm{Con}_{V_0}(S^{\mathrm{out}}Y)$$

is a polyhedral complex embedded in $\mathrm{T}M\times I$, so $\mathrm{Con}_{V_0}(M^{\mathrm{out}})\times I_t$ is a polyhedral complex embedded in $\mathrm{T}M\times I^2$. We enrich this complex by adding to it certain faces at $x=1$: first we add the image of V_0, i.e., the face $V_0(M\setminus Z(U))\times\{1\}\times I_t$ of dimension $2n+1$. We then add the $2n$-dimensional cylinders $S_pM\times\{1\}\times I_t$, one for each annulation point $p\in Z(U)$.

W obtain in this way a polyhedral complex of dimension $2n+1$:

$$\mathcal{P} = \mathrm{Con}_{V_0}(M^{\mathrm{out}})\times I \bigcup V_0(M\setminus Z(U))\times\{1\}\times I \bigcup_{p\in Z(U)} S_pM\times\{1\}\times I \subset SM\times I^2$$

If the interior of M is orientable, the $2n$-boundary of $\mathcal{P}$ is

$$\begin{aligned}\partial_{\dim(M)}\mathcal{P} &= V_0(M)\times\{1\}\times\{1\} - V_0(M)\times\{1\}\times\{0\}\\ &\quad + \sum_{p\in Z(U)} S_pM\times\{1\}\times I - \sum_{Y\in\mathcal{F}(M)} S^{\mathrm{out}}Y\times\{0\}\times I\\ &\quad + \sum_{Y\in\mathcal{F}(M)} \mathrm{Con}_{V_0}(S^{\mathrm{out}}Y)\times\{1\} - \sum_{Y\in\mathcal{F}(M)} \mathrm{Con}_{V_0}(S^{\mathrm{out}}Y)\times\{0\}.\end{aligned} \tag{6.7}$$

In order to compute the integral on M of the Pfaffian of ∇, we analyze the Pfaffian of the connection $\mathcal{D}$. It is a closed form on $SM \times I^2$, hence by the Stokes formula on polyhedral complexes (Lemma 4.9), $\int_{\partial\mathcal{P}} \mathrm{Pf}(R^{\mathcal{D}}) = 0$. Moreover, $\mathrm{Pf}(R^{\mathcal{D}})$ vanishes identically on three of the types of faces of $\partial_{2n}\mathcal{P}$ from (6.7): It vanishes on $\mathrm{Con}_{V_0}(S^{\mathrm{out}}Y)\times\{0\}$ for $Y \in \mathcal{F}(M)$ and on $V_0(M)\times\{1\}\times\{0\}$ because at $t = 0$ the connection $\mathcal{D}$ admits a parallel section s. It also vanishes on $\mathrm{Con}_{V_0}(S^{\mathrm{out}}Y)\times\{1\}$ for $Y \in \mathcal{F}(M)$ because along $\{t = 1\}$ the connection $\mathcal{D}$ is the pull-back of ∇ from the base via the projection $SM\times I \to M$, hence by functoriality the Pfaffian $\mathrm{Pf}(R^{\mathcal{D}})$ is a horizontal form. However, since $\dim(Y) < 2n = \mathrm{rk}(\mathrm{T}M)$, the Pfaffian of ∇ vanishes on Y.

In conclusion, by integrating $\mathrm{Pf}(R^{\mathcal{D}})$ on the polyhedral complexes from (6.7), we get after using (6.4) and pull-back by V_0:

$$\int_M \mathrm{Pf}(R^{\nabla}) = (2\pi)^n \chi(M) + \sum_{Y\in\mathcal{F}(M)} \int_{S^{\mathrm{out}}Y\times I_t} \mathrm{Pf}(R^{\mathcal{D}}).$$

To conclude the proof, we note that the restriction of $\mathcal{D}$ to $\{x = 0\}$ coincides with the connection D defined in (5.3). Moreover, the computation of $\int_{S^{\mathrm{out}}Y\times I} \mathrm{Pf}(R^D)$, carried out in the previous section in the regular case, is local in the base Y, so it remains valid even without the regularity assumption on M.

6.3. Odd dimensions

This case follows directly from the even-dimensional case as we now explain.

Theorem 6.4. *Let (N, g) be a compact Riemannian polyhedral manifold of odd dimension $2n-1$. Then*

$$(2\pi)^n \chi(N) = \sum_{l=1}^{2n-1} \sum_{k=\lceil\frac{l-1}{2}\rceil}^{n-1} \frac{(-1)^{l-1}\pi(2k-1)!!}{(n-1-k)!(2k+1-l)!}$$
$$\cdot \sum_{Y\in\mathcal{F}^{(l)}(N)} \int_{S^{\mathrm{out}}Y} \mathcal{B}_Y\left[(R^Y - \tfrac{1}{2}g(A,A))^{n-1-k}(A^*)^{2k+1-l}\right]|\mathrm{d}g|.$$

By convention, $(-1)!! = 0! = 0!! = 1$, and the 0^{th} power of a double form is always 1.

Proof. Apply theorem 6.1 to the product manifold $M := N \times I$, where I is the interval $[0, 1]$, endowed with the product metric $h = g + \mathrm{d}t^2$. The Euler characteristics of M and N coincide. We will exploit the fact that the vertical vector field $\frac{\partial}{\partial t}$ is parallel along M, but also along $Y \times I$ for every face Y of N.

The boundary faces of M fall into two types:

- lateral faces of the form $Y \times I$, and
- top or bottom faces of the form $Y \times \{0\}$ or $Y \times \{1\}$.

The first type of faces do not contribute in the Gauss-Bonnet formula. Indeed, the curvature form $R^{Y\times I}$ and the second fundamental form $A^{Y\times I}$ of $Y \times I$ inside $N \times I$ both vanish in the direction of the parallel vector field $\frac{\partial}{\partial t}$. It follows that the Berezin integral inside the term from theorem 6.1 corresponding to the face $Y \times I$ vanishes identically.

The outer spheres in M of the second type of faces, e.g. $Y \times \{0\}$, can be described as spherical cones over the outer sphere of Y in N. More precisely, let V be an Euclidean vector space, S the unit sphere in V, V' a hyperplane in V and $A \subset S \cap V'$ a subset of S lying in a subsphere of codimension 1. Let $\{p_0, p_1\} = V'^{\perp} \cap S$, so p_0 and p_1 are diametrally opposed and A sits in the equatorial hypersphere orthogonal to p_0 and p_1. We defined in (6.5) the spherical cone of A with respect to p_0 as the union of all geodesic segments in S linking p_0 to A. The complement of the apex p_0, the open spherical cone, is isometric to a topological product $A \times [0, \pi/2)$ with the warped product metric $\cos^2(\alpha)g_A + d\alpha^2$. Note for later use that the volume densities induced by g and g_A on the fibers of the outer spheres satisfy the identity

$$|dg| = \cos(\alpha)^{\dim(V)-2} d\alpha |dg_A|. \tag{6.8}$$

Let $S^{\mathrm{out}}(Y)$ be the outer sphere of Y in N, and $S^{\mathrm{out}}(Y \times \{0\})$ the outer sphere of $Y \times \{0\}$ in $M = N \times I$. Then for every $x \in Y$, $S^{\mathrm{out}}_x(Y \times \{0\})$ is the spherical cone with base $S^{\mathrm{out}}_x(Y)$ and apex ∂_t. Moreover, the union for all $x \in Y$ of the open spherical cones form a locally trivial bundle with fiber type $[0, \pi/2)$ over $S^{\mathrm{out}}(Y)$

We can thus carry out the integral in α (i.e., along the fibers of the spherical cone fibration) of the integrands from theorem 6.1. For a fixed $x \in Y$, the curvature R^Y and the metric contraction of the second fundamental form $g(A, A)$ are pull-backs from the base, i.e., they are constant on the outer sphere. The dual A^* of the second fundamental form is linear on the normal bundle to Y and vanishes at the apex ∂_t, hence for $v \in S^{\mathrm{out}}_x Y$,

$$A^*(\cos(\alpha)v + \sin(\alpha)\partial_t) = \cos(\alpha)A^*(v).$$

Now the volume form on the spherical cones is given by (6.8). It follows that the push-forward along the fibers of the spherical cones $S^{\mathrm{out}}_x(Y \times \{0\}) \to S^{\mathrm{out}}_x Y$ (i.e., the integral in $\alpha \in [0, \pi/2)$) of

$$\mathcal{B}_Y\left[(R^Y - \tfrac{1}{2}g(A, A))^{n-1-k}(A^*)^{2k+1-l}\right]|dg_M|$$

amounts to

$$I_{2k}\mathcal{B}_Y\left[(R^Y - \tfrac{1}{2}g(A, A))^{n-1-k}(A^*)^{2k+1-l}\right]|dg_N|$$

where I_{2k} is a scaling factor independent of x:

$$I_{2k} = \int_{-\pi/2}^{\pi/2} \cos^{2k}(\alpha)d\alpha = \frac{\pi(2k-1)!!}{(2k)!!}. \qquad \blacksquare$$

7. Constant-curvature polyhedral manifolds with geodesic faces

By applying the Gauss-Bonet theorems 6.1 and 6.4 to the case of polyhedral manifolds of constant sectional curvature with totally geodesic faces, we obtain certain identities for spherical, euclidean and hyperbolic polyhedra in terms of volumes of faces and measures of their outer angles.

7.1. Euclidean polyhedra

Let M be a flat compact polyhedral manifold of dimension k with totally geodesic faces. In this case, the Gauss-Bonnet simply states that the sum of the outer angles at the vertices of M equals the Euler characteristic $\chi(M)$ divided by the volume of the $k-1$ sphere. Indeed, in Theorems 6.1 and 6.4 the curvature R^Y of the face Y and the second fundamental form A of $Y \subset M$ both vanish, so the only non-zero terms in the right-hand side arises for $\operatorname{codim}(Y) = k$, i.e., when Y is a point. In that case, the integral corresponding to a vertex Y gives the volume of the outer sphere at the vertex Y, and the formula becomes

$$\operatorname{vol}(S^{k-1})\chi(M) = \sum_{Y \in \mathcal{F}^{(k)}} \angle^{\text{out}} Y.$$

In particular, the Euler characteristic of a flat compact polyhedral manifold with totally geodesic faces is always non-negative, and it is necessarily positive as soon as M has at least one vertex. This identity is clear for open polytopes in $\mathbb{R}^k$, since the outer spheres of the vertices partition the unit sphere S^{k-1} into spherical polytopes with mutually disjoint interiors. But in general it is not obvious. A direct proof should rely on some additivity property of outer angles.

7.2. Manifolds of constant sectional curvature with geodesic faces

Let (M, g) be a compact polyhedral manifold with constant scalar curvature $\mathfrak{k}$ and with geodesic faces. Then $R = \frac{\mathfrak{k}}{2} g^2$, valid on every face. Since the second fundamental form A of any face Y is assumed to vanish, we also have $A^* = 0$. Therefore Theorems 6.1 and 6.4 give

$$(2\pi)^n \chi(M) = \sum_{k=0}^{n} \tfrac{2^{k-1}(k-1)!}{(n-k)!} \sum_{Y \in \mathcal{F}^{2k}(M)} \int_{S^{\text{out}} Y} \mathcal{B}_Y \left[(R^Y)^{n-k} \right] |\mathrm{d}g| \qquad \text{for } \dim(M) = 2n,$$

$$2^n \pi^{n-1} \chi(N) = \sum_{k=1}^{n} \tfrac{(2k-3)!!}{(n-k)!} \sum_{Y \in \mathcal{F}^{2k-1}(N)} \int_{S^{\text{out}} Y} \mathcal{B}_Y \left[(R^Y)^{n-k} \right] |\mathrm{d}g| \qquad \text{for } \dim(N) = 2n-1$$

where we recall that $\mathcal{F}^{(h)}$ denotes the set of faces of codimension $h \geq 0$. On a face Y of dimension $2j$, we compute moreover:

$$(R^Y)^j = \mathfrak{k}^j \frac{(2j)!}{2^j} \mathrm{d}g_Y \otimes \mathrm{d}g_Y, \qquad \mathcal{B}_Y((R^Y)^j) = \mathfrak{k}^j \frac{(2j)!}{2^j} |\mathrm{d}g_Y|.$$

In conclusion, regardless of the parity of $d = \dim(M)$, the Gauss-Bonnet formula becomes the sum (1.1) over the even-dimensional faces of M^d advertised in the introduction as Theorem 1.1.

Dehn [16] studied such identities for small dimensions and predicted their existence in general. Allendoerfer-Weil's formula from [4] was used by Santaló [23] for deducing particular cases of (1.1) for polyhedra embedded in a constant curvature space-form. Peter McMullen [19] derived certain quadratic angle-sum identities for euclidean polyhedral cones by linking them to the Gauss-Bonnet formula for spherical polytopes. For a spherical simplex inside S^k, the identity was also announced by Kenzi Sato [25].

7.3. Hyperbolic polyhedra with ideal vertices

As an extension of the previous example, the hyperbolic identity allows us to compute the volume of hyperbolic $2n$-polyhedra with some – or all – ideal vertices in terms of outer angles and volumes of lower-dimensional faces, by passing to the limit the Gauss-Bonnet formula for compact polyhedra. For instance, when $2n = 4$ the volume of an ideal hyperbolic 4-simplex is given by

$$\operatorname{vol}(M) = -2\pi^2 + \frac{\pi}{3} \sum_{Y \in \mathcal{F}^{(2)}(M)} \angle^{\text{out}}(Y)$$

where $\angle^{\text{out}}(Y)$ is the outer dihedral angle of the ideal triangle Y in M, i.e., the angle between the outer normals to the two hyperfaces containing Y.

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Sergiu Moroianu
Institutul de Matematică al Academiei Române, P.O. Box 1-764, 014700 Bucharest, Romania;
moroianu@alum.mit.edu