CONVEXITY OF THE RENORMALIZED VOLUME OF HYPERBOLIC 3-MANIFOLDS

SERGIU MOROIANU

ABSTRACT. The Hessian of the renormalized volume of geometrically finite hyperbolic 3-manifolds without rank-1 cusps, computed at the hyperbolic metric g_{geod} with totally geodesic boundary of the convex core, is shown to be a strictly positive bilinear form on the tangent space to Teichmüller space. The metric g_{geod} is known from results of Bonahon and Storm to be an absolute minimum for the volume of the convex core. We deduce the strict convexity of the functional volume of the convex core at its minimum point.

1. INTRODUCTION

The renormalized volume is a functional on the moduli space of hyperbolic 3-manifolds of finite geometry. It has been introduced in this context by Krasnov [9], after initial work by Henningson and Skenderis [8] for more general Poincaré-Einstein manifolds. As 3-dimensional geometrically finite 3-manifolds are closely related to Riemann surfaces, Vol_R defines in a natural way a Kähler potential for the Weil-Petersson symplectic form on the Teichmüller space. This follows for quasi-fuchsian manifolds by the identity between the renormalized volume and the so-called classical Liouville action functional, a topological quantity known by work of Takhtadzhyan and Zograf [14] to provide a Kähler potential. For geometrically finite hyperbolic 3-manifolds without rank-1 cusps, the Kähler property of the renormalized volume was proved by Colin Guillarmou and the author in [6], by constructing a Chern-Simons theory on the Teichmüller space. The case of cusps of rank 1 is studied in a joint upcoming paper with Guillarmou and Frédéric Rochon.

Here we look at a certain moduli space of complete, infinite-volume hyperbolic metrics g on a fixed 3-manifold X. The metrics we consider are geometrically finite quotients $\Gamma\backslash\mathbb{H}^3$ (i.e., they admit a fundamental polyhedron with finitely many faces) and do not have cusps of rank 1, in the sense that every parabolic subgroup of Γ, if any, must have rank 2. We define the moduli space $\mathcal{M}$ as the quotient of the above set of metrics on X by the group $\mathrm{Diff}^0(X)$ of diffeomorphisms isotopic to the identity. The existence of such metrics on X implies that X is diffeomorphic to the interior of a manifold-with-boundary K. Let $2K$ be the smooth manifold obtained by doubling K across Σ. We make the following assumption throughout the paper:

Date: February 15, 2017.

Partially supported by the CNCS project PN-II-RU-TE-2012-3-0492.

There exists on $2K$ a complete hyperbolic metric of finite volume.

It follows from Mostow-Prasad rigidity that up to a diffeomorphism of $2K$ isotopic to the identity, the boundary $\Sigma = \partial K$ is totally geodesic for this metric. Since $2K$ must be aspherical and atoroidal, the connected components of Σ cannot be spheres or tori.

Examples of manifolds where our assumption is *not* fulfilled are quasi-fuchsian manifolds and Schottky manifolds, since their double is not atoroidal. With the above assumption, a distinguished point g_{geod} in $\mathcal{M}$ is obtained from K by gluing infinite-volume *funnels* with vanishing Weingarten operator (see Section 2) to each boundary component of K. We call this metric the *totally geodesic metric*, and note that K is the convex core of (X, g_{geod}). It was remarked by Thurston, again as a simple consequence of Mostow rigidity, that g_{geod} is the unique metric in $\mathcal{M}$ with smooth boundary of the convex core.

By work of Bonahon [3] it is known that the volume of the convex core $\mathrm{Vol}(C(X, g))$ has a minimum at g_{geod} when viewed as a functional on $\mathcal{M}$. When X is convex co-compact, i.e., without cusps, Storm [13] proved that the minimum point g_{geod} is strict. We shall apply here our results on Vol_R to deduce the convexity of $\mathrm{Vol}(C(X, g))$ at this special point in $\mathcal{M}$ for X geometrically finite without cusps of rank 1, but possibly with cusps of rank 2 as in [3].

It is instructive to compare those results to the situation for quasi-fuchsian manifolds. Combining results of Schlenker [12] and Brock [4], the renormalized volume of quasi-fuchsian manifold is commensurable on Teichmüller space to the volume of the convex core. In particular, it is *not* proper as a function on Teichmüller space, since it remains bounded under iterations of a Dehn twist. It has been stated without proof by Krasnov and Schlenker [10] that Vol_R is non-negative on the quasi-fuchsian space. There is some compelling evidence for this claim: it was proved in [10] that the only critical point in a Bers slice is at the fuchsian locus, and there the Hessian of the renormalized volume equals a multiple of the Weil-Petersson scalar product. However, the lack of properness does not allow one to conclude that Vol_R is globally non-negative. In a recent joint paper with Corina Ciobotaru, we proved that Vol_R is non-negative on the *almost-fuchsian* space, an open neighborhood of the fuchsian locus inside the quasi-fuchsian space, and that it vanishes there only at the fuchsian locus.

Schlenker's results from [12] have been recently extended to convex co-compact hyperbolic 3-manifolds by Bridgeman and Canary [1]. They obtain quite nice global results bounding the renormalized volume in terms of the convex core.

The main result of this paper describes the local behavior of Vol_R near g_{geod}.

Theorem 1. *Let g_{geod} be a geometrically finite hyperbolic metric on X without rank 1-cusps and with totally geodesic boundary of the convex core. Then the Hessian of the renormalized volume functional on $\mathcal{M}$ at g_{geod} is positive definite.*

The proof is done in two steps. First we look at the volume enclosed by minimal surfaces near the boundary of the convex core, proving that it is convex, and then compare it to the "optimal" renormalization with respect to the unique hyperbolic metric in the conformal class at infinity. In the first step we use a boundary-value problem for the linearized Einstein equation in Bianchi gauge at a metric with geodesic boundary. The

Hessian of the volume appears as a Dirichlet-to-Neumann operator, which we prove to be strictly positive by an appropriate Weitzenböck formula. The second step uses the analysis of the uniformizing conformal factor, together with some elementary elliptic theory.

As a consequence, we obtain the convexity of the convex core volume functional:

Theorem 2. *Let (X, g_{geod}) be a geometrically finite hyperbolic 3-manifold with totally geodesic boundary of the convex core, and without rank-1 cusps. Then the functional*

$$\mathcal{M} \longrightarrow \mathbb{R}, \qquad\qquad g \longmapsto \mathrm{Vol}(C(X, g))$$

is strictly convex at g_{geod}.

We do not know whether $\mathrm{Vol}(C(X, g))$ is twice differentiable as a function of g, see [3]. By strict convexity we mean that in a neighborhood of g_{geod}, $\mathrm{Vol}(C(X, g))$ is bounded from below by a smooth functional whose Hessian is positive definite as a bilinear form on $T_{g_{\mathrm{geod}}}\mathcal{M}$, and taking the same value at g_{geod} as $\mathrm{Vol}(C(X, g_{\mathrm{geod}}))$.

By the simultaneous uniformization result of Ahlfors and Bers valid for quasi-fuchsian manifolds, extended by Marden [11] to the geometrically finite case, $\mathcal{M}$ is identified with the Teichmüller space of Σ, keeping in mind that the connected components of Σ have genus at least 2. We identify therefore $T\mathcal{M}$ with the finite-dimensional space $T\mathcal{T}_\Sigma$. Hence Theorem 1 asserts that the Hessian of the renormalized volume is positive in every direction of $\mathcal{T}_\Sigma$ at the point corresponding to the totally geodesic metric on X.

Acknowledgments. This paper originated from discussions with Colin Guillarmou and Jean-Marc Schlenker about the renormalized volume of hyperbolic 3-manifolds. I owe in particular to Jean-Marc the observation that Theorem 1 has implications about the volume of the convex core. I am indebted to Corina Ciobotaru for some remarks on the manuscript, and to an anonymous referee for pointing out several inadvertencies in the initial text. It is a pleasure to acknowledge their help.

2. FUNNELS

Let (X, g) be a geometrically finite hyperbolic 3-manifold without rank-1 cusps. Such a manifold can be decomposed in a finite-volume part K (a smooth manifold-with-boundary with a finite number of cusps of rank 2), and a finite number of funnels. These funnels play an important role in this paper, so we review them below.

A *funnel* is a hyperbolic half-cylinder (F, g), where $F = [0, \infty) \times \Sigma$, for some compact, possibly disconnected Riemannian surface (Σ, h), while

$$g = dt^2 + h_t, \qquad\qquad h_t = h\left((\cosh t + A \sinh t)^2 \cdot, \cdot\right). \tag{1}$$

Here A is a symmetric field of endomorphisms of $T\Sigma$, namely the Weingarten operator of the isometric inclusion $\{0\} \times \Sigma \hookrightarrow F$. The tensors h_t are Riemannian metrics on Σ whenever t is such that the eigenvalues of A are larger than $-\coth t$. Hence, we must assume that $A + 1$ is positive definite in order for g to be well-defined on the whole half-cylinder. For notational simplicity, we allow disconnected funnels.

The necessary and sufficient conditions for g to be hyperbolic are the hyperbolic version of the Gauss and Codazzi–Mainardi equations:

$$\det(A) = \kappa_h + 1, \tag{2}$$

$$(d^\nabla)^* A + d\mathrm{Tr}(A) = 0 \tag{3}$$

where κ_h is the Gaussian curvature of h. Let

$$H := \mathrm{Tr}(A) : \Sigma \to \mathbb{R}$$

be the mean curvature function (without the customary $1/2$ factor) of $\{0\} \times \Sigma \hookrightarrow X$ with respect to the direction ∂_t escaping from K. Let A_t, H_t, κ_{h_t} be the Weingarten map, the mean curvature, respectively the Gaussian curvature, of $\{t\} \times \Sigma \hookrightarrow F$. We have

$$A_t = \tfrac{1}{2} g^{-1}\partial_t g = (\cosh t + A \sinh t)^{-1}(\sinh t + A \cosh t)$$

The Gauss and Codazzi-Mainardi equations continue to hold at every t, so

$$\kappa_{h_t} = \det(A_t) - 1.$$

3. Renormalized volumes

For a geometrically finite metric without rank-1 cusps with a fixed funnel structure, we define the induced metric at infinity on the surface Σ:

$$h_\infty := \lim e^{-2t} h_t = \tfrac{1}{4} h\left((1+A)^2 \cdot, \cdot\right). \tag{4}$$

The renormalized volume of (X, g) with respect to h_∞ is defined by Krasnov and Schlenker [10] as

$$\mathrm{Vol}_R(X, g; h_\infty) = \mathrm{Vol}(K, g) - \tfrac{1}{4} \int_\Sigma H d\mathrm{vol}_h$$

where H is the trace of A. This is the same as the Riesz-regularized volume with respect to the boundary-defining function e^{-t}, see e.g. [7]. A proof of this equality appears for instance explicitly in [5, Prop. 5]. Let $\omega \in C^\infty(\Sigma)$ be the unique conformal factor such that the metric $e^{2\omega} h_\infty$ is of constant curvature equal to -4. Like every metric in the conformal class of h_∞, the metric $e^{2\omega} h_\infty$ arises as the metric at infinity for some other funnel structure on (X, g). The *renormalized volume of* (X, g) is defined (cf. Krasnov [9]) with respect to this canonical choice:

$$\mathrm{Vol}_R(X, g) := \mathrm{Vol}_R(X, g; e^{2\omega} h_\infty).$$

We stress that the chosen metric at infinity $e^{2\omega} h_\infty$ is not hyperbolic, but rather of curvature -4, so the area of $(M, e^{2\omega} h_\infty)$ equals $\pi(g-1)$.

From [5, Lemma 7 and 8], for every other metric h' conformal to h_∞ and of area less than or equal to $\pi(g-1)$, we have $\mathrm{Vol}_R(X, g; h') < \mathrm{Vol}_R(X, g; e^{2\omega} h_\infty)$, hence the above choice is very natural.

4. GEOMETRICALLY FINITE MANIFOLDS WITH TOTALLY GEODESIC BOUNDARY

Let g be a hyperbolic metric on X such that the convex core of X has totally geodesic boundary, denoted Σ. Then $A = 0$, $H = 0$, h is hyperbolic and the induced metric at infinity

$$h_\infty = \lim e^{-2t} h_t = h/4$$

has constant Gaussian curvature equal to -4. Let $\{g^s\}_{s\in\mathbb{R}}$ be a one-parameter family of deformations of g inside the space of geometrically finite hyperbolic metrics on X. (This space is parametrized by the deformations $[h^s_\infty]$ of the conformal classes of the induced metrics h^s_∞ at infinity).

Proposition 3. *For small deformations g^s of g, in the homotopy class of Σ there exists a unique family of surfaces Σ_s which are minimal for g^s.*

Proof. For each connected component Σ_j of Σ cut out the funnel containing it and complete it to a quasi-fuchsian manifold (this is possible since for small enough s, the eigenvalues of A_s are close to 0). In that quasi-fuchsian manifold it is known e.g. by Uhlenbeck [15] that there exists a unique minimal surface homotopic to Σ_j. This surface will live in the original funnel of X for small enough s. □

Note that Σ is π_1-incompressible in the sense that for all j, $\pi_1(\Sigma_j)$ injects into $\pi_1(X)$.

For s close to 0 let therefore $\Sigma_s \subset X$ be the unique minimal surface inside X homotopic to Σ. Choose $\{\Phi_s\}$ a family of diffeomorphisms of X mapping Σ onto Σ_s, with Φ_0 equal to the identity, and furthermore such that the outgoing geodesics on Σ are mapped isometrically on the corresponding geodesics normal to Σ_s with respect to the metric g^s. Hence, by pulling back g^s via Φ_s we may assume that $[0,\infty) \times \Sigma$ is the underlying space of every funnel in the family g^s, of course with different metric h^s and Weingarten map A^s.

By composing with an additional family of diffeomorphisms preserving the surface Σ and the funnel structure, we can further assume that the first-order variation $\dot{h}$ of the metrics h^s induced by g^s on Σ (the dot on top of some tensor denotes s-derivative at $s = 0$) is divergence-free:

$$\delta^h \dot{h} = 0.$$

Let $\dot{A}$ be the first-order variation of the Weingarten map.

Lemma 4. *The tensors $\dot{h}$ and $\dot{A}$ along Σ are trace- and divergence-free.*

Proof. Since h^s is minimal, we have $\mathrm{Tr}(A^s) = 0$, and hence $\mathrm{Tr}(\dot{A}) = 0$. On one hand, differentiating the Gauss equation implies that the variation of the curvature of h^s is 0:

$$\frac{\partial}{\partial s}\kappa(h^s)_{|s=0} = \mathrm{Tr}(\dot{A}) = 0.$$

On the other hand, the variation of κ at the hyperbolic metric h is given by the following intrinsic formula (cf. e.g. [2, Theorem 1.174(e)]):

$$2\dot{\kappa} = (\Delta_h + 1)\mathrm{Tr}(\dot{h}) + d^*\delta^h \dot{h}. \tag{5}$$

Since $\delta^h \dot{h} = 0$ and $\dot{\kappa} = 0$, it follows by positivity of the elliptic operator $\Delta_h + 1$ that $\mathrm{Tr}(\dot{h}) = 0$. Differentiating the Codazzi equation $\delta^{h^s} A^s = 0$ shows, since $A = 0$, that $\delta^h \dot{A} = 0$. □

Lemma 5. *The tensor $\dot{g}$ on X is trace- and divergence-free in a neighborhood of Σ containing the funnel.*

Proof. We have

$$\dot{g} = \cosh^2(t)\dot{h} + 2\sinh(t)\cosh(t)h(\dot{A}\cdot, \cdot).$$

This tensor is clearly trace-free, since $\dot{A}, \dot{h}$ are trace-free.

Let $T := \partial_t$ (we denote by ν the restriction of T along Σ). Write $f = \cosh(t)$ so that

$$g = dt^2 + f^2 h, \qquad \dot{g} = f^2 \dot{h} + 2ff' h(\dot{A}\cdot, \cdot).$$

For every tangential vector fields U, V independent of t, i.e., such that $[T, U] = [T, V] = 0$, we have directly from the Koszul formula

$$\nabla_T T = 0, \qquad \nabla_T U = \tfrac{f'}{f} U = \nabla_U T, \qquad \nabla_U V = \nabla^\Sigma_U V - ff' h(U, V)T.$$

For a 1-form α with $L_T \alpha = 0$ and $\alpha(T) = 0$, we get by duality

$$\nabla_T dt = 0, \qquad \nabla_T \alpha = -\tfrac{f'}{f}\alpha, \qquad \nabla_U \alpha = \nabla^\Sigma_U \alpha - ff'\alpha(U)dt.$$

Since $\dot{g}(T, \cdot) = 0$ it follows from the above table that $(\nabla_T \dot{g})(T, \cdot) = 0$. Moreover, if $\{e_1, e_2\}$ is an orthonormal frame for h, then

$$-\sum_{i=1}^{2} (\nabla_{e_i}\dot{g})(e_i, e_j) = -\sum_{i=1}^{2} (\nabla^\Sigma_{e_i}\dot{g})(e_i, e_j), \qquad -\sum_{i=1}^{2} (\nabla_{e_i}\dot{g})(e_i, T) = ff'\mathrm{Tr}(\dot{g}).$$

Both these terms vanish by Lemma 4. □

5. Variation of the Einstein equation

In dimension 3, a hyperbolic metric means an Einstein metric with constant -2:

$$\mathrm{Ric} = -2g. \tag{6}$$

The first-order variation of this equation along a path of metrics reads ([2, Theorem 1.174.d]):

$$-\delta^*(\delta + \tfrac{1}{2} d\mathrm{Tr})\dot{g} + \tfrac{1}{2}\left[\nabla^*\nabla \dot{g} + \mathrm{Ric} \circ \dot{g} + \dot{g} \circ \mathrm{Ric} - 2\mathring{R}\dot{g}\right] = -2\dot{g} \tag{7}$$

where the action of the curvature tensor R on a symmetric 2-tensor h is defined as

$$(\mathring{R}h)_{iq} = \sum_{j,k=1}^{3} h_{jk}\langle R_{ij} v_k, v_q\rangle.$$

Using (6), equation (7) is equivalent to

$$-\delta^*(2\delta + d\mathrm{Tr})\dot{g} + \nabla^*\nabla\dot{g} - 2\mathring{R}\dot{g} = 0.$$

A simple computation shows that for g hyperbolic,

$$\mathring{R}h = h - \mathrm{Tr}(h)g \tag{8}$$

for every symmetric 2-tensor h.

5.1. **Weitzenböck formula for symmetric tensors.** The following Weitzenböck formulae hold on hyperbolic 3-manifolds for the rough Laplacian $\nabla^*\nabla$ on symmetric 2-tensors in terms of the twisted Hodge Laplacian $d^\nabla d^{\nabla^*} + d^{\nabla^*} d^\nabla$ on traceless symmetric 2-tensors, and of the Laplacian on functions (cf. [2]): if q_0 is a traceless symmetric 2-tensor and ψ is a smooth function, then

$$\nabla^*\nabla q_0 = (d^\nabla d^{\nabla^*} + d^{\nabla^*} d^\nabla + 3)q_0,$$
$$\nabla^*\nabla(\psi g) = \Delta(\psi)g.$$

Moreover, by (8),

$$\mathring{R}(\psi g) = -2\psi g, \qquad \mathring{R}q_0 = q_0.$$

5.2. **The Laplace equation on 1-forms.** Let $\Delta = \nabla^*\nabla$ be the rough Laplacian acting on 1-forms (equivalently, on vector fields) on the compact manifold with boundary K. Clearly, Δ maps $C^\infty(K, TK)$ to itself. Recall that ν is the unit outgoing vector field orthogonal to the boundary Σ of K, and L_ν denotes the Lie derivative.

Proposition 6. *The restriction*

$$\Delta + 2 : \{V \in C^\infty(K, TK); V_{|\Sigma} \in T\Sigma, L_\nu V \perp \Sigma\} \to C^\infty(K, TK)$$

is an isomorphism.

Proof. Let $\mathcal{D}$ denote the initial domain $\{V \in C^\infty(K, TK); V_{|\Sigma} \in T\Sigma, L_\nu V \perp \Sigma\}$. Then by integration by parts using that Σ is totally geodesic, we have for all $V, V' \in \mathcal{D}$:

$$\langle \nabla^*\nabla V, V' \rangle = \langle \nabla V, \nabla V' \rangle = \langle V, \nabla^*\nabla V' \rangle.$$

This implies that $\nabla^*\nabla$ is symmetric and non-negative on $\mathcal{D}$. Its self-adjoint Friedrichs extension

$$\Delta_{\mathcal{F}} : \mathcal{D}_{\mathcal{F}} \to L^2$$

is therefore also non-negative, so $\nabla^*\nabla + 2 : \mathcal{D}_{\mathcal{F}} \to L^2$ is invertible. By elliptic regularity, the preimage of $C^\infty(K, TK)$ by this operator must lie in $\mathcal{D}_{\mathcal{F}} \cap C^\infty(K, TK) = \mathcal{D}$. □

6. THE HESSIAN OF THE VOLUME OF COMPACT HYPERBOLIC 3-MANIFOLDS WITH GEODESIC BOUNDARY

Theorem 7. *Let (K, g) be a compact hyperbolic 3-manifold with totally geodesic boundary, and $\{g^s\}_{s\in\mathbb{R}}$ a smooth family of hyperbolic metrics on K with minimal boundary. Then the Hessian of the volume functional of K at g is positive.*

Proof. For small s, the principal curvatures along K are smaller than 1, so equation (1) defines a funnel, extending g^s to a complete hyperbolic metric on $X = K \cup F$, unique up to isometry. By hypothesis, Σ, the possibly disconnected boundary of K, is minimal for each of the metrics g^s. Moreover, the outgoing normal geodesics to Σ with respect to g are also parametrized geodesics for g^s. By composing with a family of diffeomorphisms of X preserving Σ, we can assume that $\dot{h}$, the first-order variation of the metrics g^s restricted to Σ, is divergence-free. It follows that we can apply the results of Section 4, in particular $\dot{h}$ and A are divergence-free, trace-free along Σ, while $\dot{g}$ is divergence-free, trace free on the funnel.

Consider the following boundary-value problem:

$$\begin{cases} (\nabla^*\nabla + 2)V = -(\delta + \frac{1}{2}d\mathrm{Tr})\dot{g}, \\ V \in C^\infty(K, TK), \\ V_{|\Sigma} \in T\Sigma, \\ L_\nu V \perp \Sigma. \end{cases} \tag{9}$$

By Proposition 6, there exists a unique solution V to (9). Set

$$q := \dot{g} + L_V g.$$

If $\{\phi_s\}$ is the 1-parameter group of diffeomorphisms of K integrating V (well-defined since V is tangent to ∂K), then q is the tangent vector field to the 1-parameter family of metrics $G^s := \phi_s^* g^s$.

Remark that

- G^s is hyperbolic;
- $\mathrm{Vol}(K, g^s) = \mathrm{Vol}(K, G^s)$;
- Σ is minimal in (K, G^s) for every s;
- Σ is totally geodesic at $s = 0$;
- $\nu^s := \phi^{s*}\nu$ is the unit normal vector field to Σ with respect to G^s.

The last property holds because ν is the unit normal vector field to Σ with respect to g^s for every s.

By the Schläfli formula of Rivin-Schlenker (see [7], Lemma 5.1), we have

$$\partial_s \mathrm{Vol}(K, g^s) = \frac{1}{2}\int_\Sigma (\mathrm{Tr}(\dot{A}^s) + \frac{1}{2}\mathrm{Tr}((h^s)^{-1}\dot{h}^s A^s))d\mathrm{vol}_{h^s} = \frac{1}{8}\langle \dot{h}^s, L_\nu g^s\rangle_{L^2(\Sigma, h^s)}.$$

The same formula for the family of metrics $G^s = \phi_s^* g^s$ gives

$$\partial_s \mathrm{Vol}(K, \phi_s^* g^s) = \frac{1}{8}\langle \partial_s G^s, L_{\phi^{s*}\nu} G^s\rangle_{L^2(\Sigma, G^s)}. \tag{10}$$

The term $L_{\phi^{s*}\nu} G^s$, i.e., the second fundamental form of Σ with respect to G^s, vanishes at $s = 0$. One more derivative at $s = 0$ shows therefore

$$\partial_s^2 \mathrm{Vol}(K, G^s)_{s=0} = \frac{1}{8}\langle q, L_\nu q - L_{[V,\nu]} g\rangle_{L^2(\Sigma, h)}. \tag{11}$$

Since $\mathrm{Vol}(K, g^s) = \mathrm{Vol}(K, G^s)$, the above formula computes the second variation $\ddot{\mathrm{Vol}}$ of $\mathrm{Vol}(K, g^s)$.

Theorem 8. *The inner product $\langle q, L_\nu q\rangle_{L^2(\Sigma,g)}$ is non-negative. Explicitly,*

$$\langle q, L_\nu q\rangle_{L^2(\Sigma,g)} \geq \|q\|^2.$$

Proof. The tensor q is a solution to the linearized Einstein equation because the family G^s consists of hyperbolic metrics. Use now the following identities on vector fields:

$$(2\delta + d\mathrm{Tr})\delta^* = \nabla^*\nabla + 2, \qquad\qquad 2\delta^* V = L_V g.$$

From (9), it follows that $q = \dot{g} + L_V g$ is in Bianchi gauge, i.e.,

$$(\delta + \tfrac{1}{2} d\mathrm{Tr})q = 0.$$

Equation (7) implies that q is a solution of the elliptic equation

$$(\nabla^*\nabla - 2\mathring{R})q = 0. \tag{12}$$

Decompose q in its trace-free component q_0 and its pure trace component ψg for some $\psi \in C^\infty(K)$. Using the Weitzenböck formulae from section 5.1, (12) is equivalent to

$$\begin{cases} (d^\nabla d^{\nabla^*} + d^{\nabla^*} d^\nabla + 1)q_0 = 0, \\ (\Delta + 4)a = 0. \end{cases} \tag{13}$$

Because of these identities, integration by parts (i.e., Green's formula) on K gives

$$\int_\Sigma \langle q_0, L_\nu q_0\rangle d\mathrm{vol}_h = \int_K (|d^\nabla q_0|^2 + |d^{\nabla^*} q_0|^2 + |q_0|^2) d\mathrm{vol}_g, \tag{14}$$

$$\int_\Sigma \langle \psi, L_\nu \psi\rangle d\mathrm{vol}_h = \int_K (|d\psi|^2 + 4\psi^2) d\mathrm{vol}_g. \tag{15}$$

We have used $L_\nu q_0 = \nabla_\nu q_0$ (equivalent to $(\Sigma, h) \hookrightarrow (K, g)$ being totally geodesic). Since by the same reason $L_\nu g = 0$, (15) is the same as

$$\int_\Sigma \langle \psi g, L_\nu(\psi g)\rangle d\mathrm{vol}_h = \int_K (|d\psi|^2 + 4\psi^2) d\mathrm{vol}_g. \tag{16}$$

Since $\mathrm{Tr}(g^{-1}q_0) = 0$ by definition and $L_\nu g = 0$, it follows by applying L_ν that

$$\mathrm{Tr}(g^{-1} L_\nu q_0) = 0.$$

Hence $L_\nu q_0$ is trace-free, $L_\nu(\psi g)$ is a multiple of g, and so (14) and (16) give

$$\int_\Sigma \langle q, L_\nu q\rangle d\mathrm{vol}_h \geq \|q_0\|^2 + 4\|\psi\|^2 \geq \|q\|^2.$$

□

Returning to (11), we would like to analyze the remaining term. In fact we prove below that it vanishes pointwise on Σ, thereby ending the proof of Theorem 7. □

Proposition 9. *The scalar product $\langle q, L_{[V,T]} g\rangle_g$ is pointwise zero on Σ.*

Proof. Let
$$V = v_0 + tu_1 T + t^2(v_2 + u_2 T) + O(t^3)$$
be the Taylor expansion of V near $t = 0$, using the fixed product decomposition near Σ. Here v_0, v_2 are vector fields on Σ, while u_1, u_2 are functions. Note that the coefficients u_0 and v_1 vanish (and we omit them from the formula) as a consequence of (9).

From the definition, $q = \dot{g} + L_V g$ where we can also write $L_V g = 2(\nabla V)_{\mathrm{sym}} = 2\delta^* V$. We recall that $\dot{g}$ is tangential (it does not contain terms involving dt). The correction term equals at Σ

$$L_V g = L_{v_0} h + u_1 dt \otimes dt + O(t) \tag{17}$$

and so in particular it has no mixed terms of the type $dt \otimes \Lambda^1\Sigma$. The vector field $[T, V]$ equals
$$[T, V] = u_1 T + 2tv_2 + 2tu_2 T + O(t^2).$$
The tensor $L_{[T,V]}g$ does not have any tangential component at $t = 0$. The mixed terms do not contribute in the scalar product with q since that last tensor has no such mixed terms. The coefficient of $dt \otimes dt$ in $L_{[T,V]}g$ is $2u_2$. But this term vanishes by the lemma below. □

Lemma 10. *The second-order normal term u_2 in V vanishes.*

Proof. The free term $-2\delta\dot{g} - d\mathrm{Tr}(\dot{g})$ in the boundary-value problem (9) determining V vanishes near Σ by Lemma 5. At $t = 0$ the Hodge Laplacian $\Delta_H = dd^* + d^*d$ on 1-forms takes the form
$$(\Delta_H V)_{|t=0} = \Delta^h v_0 - 2v_2 - 2u_2\nu.$$
Since g is hyperbolic, Bochner's formula
$$\Delta_H = \nabla^*\nabla + \mathrm{Ric}$$
gives $(\Delta_H + 4)V = (\Delta + 2)V = 0$, and using that V is tangent to Σ we deduce $u_2 = 0$. □

7. The Hessian of the renormalized volume on the funnel

We have seen above that $\mathrm{vol}(K)$ has positive Hessian at g. But since Σ is minimal for g^s, we remark that
$$\mathrm{Vol}_R(X, g^s; h^s_\infty) = \mathrm{Vol}(K, g^s).$$
To prove Theorem 1 we must therefore analyze the Hessian of $\mathrm{Vol}_R - \mathrm{Vol}(K)$. For this, let $\omega^s \in C^\infty(\Sigma)$ be the conformal factor so that $e^{2\omega^s} h^s_\infty$ has constant curvature -4. Such a conformal factor is unique, smooth in s, and $\omega^0 = 0$.

Following the proof of [5, Theorem 10], one could prove the following inequality by showing that the area of h^s_∞ is at most equal to $\pi(g - 1)$:

Proposition 11. *For small s,*
$$\mathrm{Vol}_R(X, g^s) \geq \mathrm{Vol}_R(X, g^s; h^s_\infty),$$
with equality at $s = 0$.

As a consequence, $\ddot{V}_R \geq \frac{d^2}{ds^2}\mathrm{Vol}_R(X, g^s; h^s_\infty)_{|s=0}$. However, rather than adapting the results of [5], we prove below that the functional $\mathrm{Vol}_R - \mathrm{Vol}(K)$ is convex at $g = g_{\mathrm{geod}}$, i.e., we obtain a positive-definite lower bound for the Hessian.

The following Polyakov-type formula holds for the conformal variation of the renormalized volume (cf. [7]):

$$\mathrm{Vol}_R(X, g^s; e^{2\omega^s} h^s_\infty) - \mathrm{Vol}_R(X, g^s; h^s_\infty) = -\tfrac{1}{4}\int_\Sigma \left(|d\omega^s|^2_{h^s_\infty} + 2\kappa_{h^s_\infty}\omega^s\right) d\mathrm{vol}_{h^s_\infty}. \tag{18}$$

Let $\ddot{\kappa}$ be the second variation of $\kappa_{h^s_\infty}$ at $s = 0$:

$$\ddot{\kappa} = \partial_s^2(\kappa_{h^s_\infty})_{|s=0}.$$

Lemma 12. *Let $\omega^s \in C^\infty(\Sigma)$ such that $\kappa_{e^{2\omega^s}h^s_\infty} = -4$. Then $\omega^s = \omega_2 s^2 + O(s^3)$ with*

$$\omega_2 = -\tfrac{1}{2}(\Delta_{h^0_\infty} + 8)^{-1}\ddot{\kappa}.$$

Proof. The metrics h^s are given by

$$h^s_\infty = \tfrac{1}{4}h^s\left((1 + A^s)^2\cdot, \cdot\right)$$

hence the first-order variation is

$$\dot{h}_\infty = \tfrac{1}{4}(\dot{h} + 2h(\dot{A}\cdot, \cdot)). \tag{19}$$

By applying the formula (5), we get for the first-order variation of the Gaussian curvature of h^s_∞:

$$2\dot{\kappa}_\infty = (\Delta_\infty + 1)\mathrm{Tr}(\dot{h}_\infty) + d^*\delta\dot{h}_\infty.$$

Both $\mathrm{Tr}(\dot{h}_\infty)$ and $\delta\dot{h}_\infty$ vanish by Lemma 4, thus $\dot{\kappa}_\infty = 0$.

By the conformal change rule for the Gaussian curvature,

$$-4 = \kappa_{e^{2\omega^s}h^s_\infty} = e^{-2\omega^s}(\kappa_{h^s_\infty} + \Delta_{h^s_\infty}\omega^s). \tag{20}$$

Let $\omega^s = s\omega_1 + s^2\omega_2 + O(s^3)$ be the limited Taylor expansion of ω. Write the expansion in s up to errors of order s^3 of the above identity, using $\kappa_{h^s_\infty} = -4 + O(s^2)$:

$$-4 = (1 - 2\omega_1 s + O(s^2))\left(-4 + O(s^2) + (\Delta_{h^0_\infty} + O(s))(s\omega_1 + O(s^2))\right)$$

giving for the coefficient of s

$$\Delta_{h^0_\infty}\omega_1 + 8\omega_1 = 0.$$

It follows by positivity of this elliptic operator on the closed surface Σ that $\omega_1 = 0$, and returning to (20),

$$-4 = (1 - 2\omega_2 s^2 + O(s^3))\left(-4 + \tfrac{1}{2}\ddot{\kappa}s^2 + O(s^3) + (\Delta_{h^0_\infty} + O(s))(s^2\omega_2 + O(s^3))\right).$$

The coefficient of s^2 must be 0, hence

$$8\omega_2 + \tfrac{1}{2}\ddot{\kappa} + \Delta_{h^0_\infty}\omega_2 = 0,$$

proving the lemma. ∎

Using this lemma, (18) gives for the quadratic term in the right-hand side:

$$-\tfrac{1}{4}\int_\Sigma \left(|d\omega^s|^2_{h^s_\infty} + 2\kappa_{h^s_\infty}\omega^s\right) d\mathrm{vol}_{h^s_\infty} = -s^2\int_\Sigma (\Delta_{h^0_\infty}+8)^{-1}\ddot{\kappa} d\mathrm{vol}_{h^s_\infty} + O(s^3).$$

By decomposing $\ddot{\kappa}$ in eigenmodes for $\Delta_{h^0_\infty}$, we see that in the integral the only surviving such term is the zero-eigenmode, hence

$$\mathrm{Vol}_R(X, g^s; e^{2\omega^s}h^s_\infty) - \mathrm{Vol}_R(X, g^s; h^s_\infty) = -\tfrac{1}{8}s^2\int_\Sigma \ddot{\kappa} d\mathrm{vol}_{h^s_\infty} + O(s^3). \tag{21}$$

Lemma 13. *The second variation of* $\kappa_{h^s_\infty}$ *equals* $-8\mathrm{Tr}(\dot{A}^2)$.

Proof. We have (see [5], proof of Lemma 10):

$$\kappa_{h^s_\infty} = \frac{4\kappa_{h^s}}{2+\kappa_{h^s}} = 4 - \frac{8}{2+\kappa_{h^s}}.$$

Thus, since $\partial_s\kappa_{h^s}|_{s=0} = 0$, we get $\partial_s^2\kappa_{h^s_\infty}|_{s=0} = 8\ddot{\kappa}_{h^s}|_{s=0}$.

From the Gauss equation (2) and the fact that $\mathrm{Tr}(A^s) = 0$ we deduce

$$\mathrm{Tr}((A^s)^2) = -2\kappa_{h^s} - 2.$$

Since $A = 0$ at $s = 0$, we get $\ddot{\kappa}_{h^s}|_{s=0} = -\mathrm{Tr}(\dot{A}^2)$. □

8. Proof of Theorem 1

From (21), Theorem 8 and Lemma 13, we get

$$\ddot{\mathrm{Vol}}_R(X, g^s) \geq \tfrac{1}{4}\int_\Sigma \mathrm{Tr}(\dot{A}^2) d\mathrm{vol}_h + \int_K \|q\|^2 d\mathrm{vol}_g.$$

First note that the right-hand side is non-negative. Indeed, by differentiating the identity

$$h^s(A^s\cdot, \cdot) = h^s(\cdot, A^s\cdot)$$

at $s = 0$ and using the fact the $A^0 = 0$ (i.e., Σ is totally geodesic for g^0), we obtain that $\dot{A}$ is symmetric, thus $\mathrm{Tr}(\dot{A}^2) \geq 0$, with equality if and only if $\dot{A} = 0$.

Secondly, assume $\ddot{\mathrm{Vol}}_R(X, g^s) = 0$. This implies that $\dot{A} = 0$ and $q = 0$. If $q = 0$, it follows that $\dot{h} + L_V h = 0$ as a tensor on Σ. Since $\dot{h}$ is divergence-free, the tensors $\dot{h}$ and $L_V h$ are orthogonal in L^2, so they both must vanish. Hence in the equality case, both $\dot{A}$ and $\dot{h}$ vanish. Then (4) shows that $\dot{h}_\infty = 0$. Since the deformation space of geometrically finite hyperbolic metrics on X is diffeomorphic to the Teichmüller space of the ideal boundary, it follows that $\dot{g}$ is a trivial deformation, ending the proof.

Remark 14. On the quasi-fuchsian space, the Hessian of Vol_R at any point of the fuchsian locus equals

$$\ddot{\mathrm{Vol}}_R = \tfrac{1}{8}\|\dot{h}^+ - \dot{h}^-\|^2,$$

Thus the Hessian is only positive *semi*definite in that case. It becomes however positive definite when we restrict the renormalized volume functional to a Bers slice, i.e., we keep fixed one conformal boundary. In that case it equals $1/8$ of the Weil-Petersson metric, by a result of Krasnov and Schlenker [10].

9. Proof of Theorem 2

Schlenker [12, Theorem 1.1] proved the following inequality between the renormalized volume and the volume of the convex core of quasi-fuchsian manifolds:

$$\mathrm{Vol}_R(X, g^s) \leq \mathrm{Vol}(C(X, g^s)) - \tfrac{1}{4}L_m(l) \tag{22}$$

where $L_m(l) \geq 0$ is the length of the measured bending lamination of the convex core. The proof consists in identifying the right-hand side with the renormalized volume of X starting from the boundary of the convex core. The induced metric at infinity is Thurston's grafting metric on Σ, which is known to be of class $C^{1,1}$, of non-positive curvature, and bounded from below by the Poincaré (i.e., hyperbolic) metric on Σ. These properties imply (22).

This inequality carries over with the same proof to convex co-compact manifolds, as was noted in [1]. In fact, Schlenker's proof remains valid for manifolds with funnels and with rank 2-cusps (i.e., geometrically finite without rank 1-cusps). Hence we may safely use (22) in our setting.

At the initial metric g_{geod} the two sets K and $C(X, g_{\mathrm{geod}})$ are the same, in particular they share the same volume. Moreover, $\mathrm{Vol}_R(X, g_{\mathrm{geod}}) = \mathrm{Vol}(K)$ since the boundary of K is totally geodesic with respect to g_{geod}. Since by Theorem 1 the Hessian of $\mathrm{Vol}_R(X, g)$ is positive definite at g_{geod}, the inequality (22) together with the positivity of $L_m(l)$ shows that the functional $\mathrm{Vol}(C(X, g))$ is bounded from below by a non-negative, smooth functional which is strictly convex at g_{geod}.

References

[1] M. Bridgeman, R. Canary, *Renormalized volume and the volume of the convex core*, arXiv:1502.05018.

[2] A. Besse, *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete **10**, Springer, 1987.

[3] F. Bonahon, *A Schläfli-type formula for convex cores of hyperbolic 3-manifolds*, J. Differential Geom. **50** (1998), 25–58.

[4] J. Brock, *The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores*, J. Amer. Math. Soc. **16** (2003), no. 3, 495–535 (electronic).

[5] C. Ciobotaru, S. Moroianu, *Positivity of the renormalized volume of almost-Fuchsian hyperbolic 3-manifolds*, Proc. AMS **144** (2016), 151–159.

[6] C. Guillarmou, S. Moroianu, *Chern-Simons line bundle on Teichmüller space,* Geometry & Topology **18** (2014), 327–377.

[7] C. Guillarmou, S. Moroianu, J.-M. Schlenker, *The renormalized volume and uniformization of conformal structures*, to appear in J. Inst. Math. Jussieu, doi: 10.1017/S1474748016000244.

[8] M. Henningson, K. Skenderis, *The holographic Weyl anomaly*, J. High Energy Phys. (1998), no. 7, paper 23, 12 pp. (electronic).

[9] K. Krasnov, *Holography and Riemann surfaces*, Adv. Theor. Math. Phys. **4** (2000), no. 4, 929–979.

[10] K. Krasnov, J.-M. Schlenker, *On the Renormalized Volume of Hyperbolic 3-Manifolds*, Commun. Math. Phys. **279** (2008), no. 3, 637–668.

[11] A. Marden, *Deformation of Kleinian groups*, Chap. 9, Handbook of Teichmüller theory, Vol. I, ed. A. Papadopoulos, IRMA Lect. Math. Theor. Phys. 11, Eur. Math. Soc. (2007).

[12] J.-M. Schlenker, *The renormalized volume and the volume of the convex core of quasifuchsian manifolds*, Math. Res. Lett. **20** (2013), 773–786.
[13] P. Storm, *Hyperbolic convex cores and simplicial volume*, Duke Math. J. **140** (2007), 281-319.
[14] L. A. Takhtadzhyan, P. G. Zograf, *On the uniformization of Riemann surfaces and on the Weil-Petersson metric on the Teichmüller and Schottky spaces*, Mat. Sb. (N.S.) **132**(174) (1987), no. 3, 304–321, 444; translation in Math. USSR-Sb. **60** (1988), no. 2, 297–313.
[15] K. Uhlenbeck, *Closed minimal surfaces in hyperbolic 3-manifolds*, Seminar on minimal submanifolds, Ann. Math. Stud. **103**, Princeton Univ. Press, Princeton, NJ., 147–168, 1983.

SERGIU MOROIANU, INSTITUTUL DE MATEMATICĂ AL ACADEMIEI ROMÂNE, P.O. BOX 1-764, RO-014700 BUCHAREST, ROMANIA

E-mail address: `moroianu@alum.mit.edu`