ON THE SINGULARITIES OF THE ZETA AND ETA FUNCTIONS OF AN ELLIPTIC OPERATOR

PAUL LOYA, SERGIU MOROIANU, AND RAPHAËL PONGE

Abstract. Let P be a selfadjoint elliptic operator of order $m > 0$ acting on the sections of a Hermitian vector bundle over a compact Riemannian manifold of dimension n. General arguments show that its zeta and eta functions may have poles only at points of the form $s = \frac{k}{m}$, where k ranges over all non-zero integers $\leq n$. In this paper, we construct elementary and explicit examples of perturbations of P which make the zeta and eta functions become singular at *all* points at which they are allowed to have singularities. We proceed within three classes of operators: Dirac-type operators, selfadjoint first-order differential operators, and selfadjoint elliptic pseudodifferential operators. As consequences, we obtain genericity results for the singularities of the zeta and eta functions in those settings. In particular, in the setting of Dirac-type operators we obtain a purely analytical proof of a well known result of Branson-Gilkey [BG], which was obtained by invoking Riemannian invariant theory. As it turns out, the results of this paper contradict Theorem 6.3 of [Po1]. Corrections to that statement are given in Appendix B.

1. Introduction

Let $P : C^\infty(M, E) \to C^\infty(M, E)$ be a selfadjoint elliptic ΨDO of order m, $m \in \mathbb{N}$, acting on the sections of a Hermitian vector bundle E over a compact Riemannian manifold M^n. The zeta and eta functions of P are two of the most important spectral functions that can be associated to P. Not only do they allow us to study the spectral properties of P (including its spectral asymmetry), but when P is a geometric operator (e.g., a Laplace-type or Dirac-type operator) they may carry a lot of a geometric information on the manifold M.

In particular, the residues of the zeta and eta functions are of special interest. More precisely, on the one hand, the residues at integer points of the zeta functions of the square of the Dirac operators are important in the context of noncommutative geometry (see, e.g., [Ka], [Po2]). On the other hand, the residues of the eta function at integer points naturally come into in the index formula for Dirac operators on manifolds with singularities (see [BS]).

Recall that the zeta and eta functions of P are obtained as meromorphic continuations from the half-plane $\Re s > \frac{n}{m}$ to the whole complex plane of the functions,

$$\zeta(P; s) := \sum_{\lambda \in \operatorname{Sp} P \setminus \{0\}} \lambda^{-s} \qquad \text{and} \qquad \eta(P; s) := \sum_{\lambda \in \operatorname{Sp} P \setminus \{0\}} \operatorname{sign}(\lambda) |\lambda|^{-s},$$

where each eigenvalue is repeated according to multiplicity and λ^{-s} is defined in a suitable way (there are actually several possible zeta functions; see Section 2).

General considerations show that the possible singularities of the meromorphic functions $\zeta(P; s)$ and $\eta(P; s)$ may only be simple poles contained in the admissible

2000 *Mathematics Subject Classification.* Primary 58J50, 58J42; Secondary 58J40.

Key words and phrases. Eta functions, noncommutative residue, pseudodifferential operators.

S.M. was partially supported by grant PN-II-ID-PCE 1188 265/2009 (Romania). R.P. was partially supported by JSPS Grant-in-Aid 30549291 (Japan).

set,

$$\Sigma := \left\{ \frac{k}{m};\ k \in \mathbb{Z},\ k \leq n \right\} \setminus \{0\}\,. \tag{1.1}$$

It then is natural to raise the following questions:

- Given any point of the admissible set Σ, is there always an operator whose zeta or eta function is singular at that point?
- Is there an operator whose zeta and eta functions are singular at *all* the points of the admissible set Σ?
- If these phenomena do occur, how "generic" are they?

In fact, if we endow the space $\Psi^m(M,E)$ with its standard Fréchet space topology, then we may expect that, *generically*, the zeta and eta function are singular at *all* the points of the admissible set Σ. To see this, let us denote by $\Psi^m(M,E)^{\mathrm{sa}}$ the real vector space of selfadjoint elements of $\Psi^m(M,E)$ equipped with the induced topology, and consider a small open neighborhood $\mathcal{W}$ of the origin in $\Psi^m(M,E)^{\mathrm{sa}}$ such that $P+R$ is elliptic for all $R \in \mathcal{W}$. For $\sigma \in \Sigma$ define

$$\mathcal{W}_\sigma := \{R \in \mathcal{W};\ \mathrm{Res}_{s=\sigma}\, \zeta(P;s) \neq 0 \text{ and } \mathrm{Res}_{s=\sigma}\, \eta(P+R;s) \neq 0\}\,.$$

Because the conditions $\mathrm{Res}_{s=\sigma}\, \zeta(P;s) \neq 0$ and $\mathrm{Res}_{s=\sigma}\, \eta(P+R;s) \neq 0$ can be shown to be open conditions and reduce to simple conditions on the homogeneous components of the symbol of R (if, say, R is supported in a local chart), we may certainly expect $\mathcal{W}_\sigma$ to be a dense open subset of $\mathcal{W}$. The Baire category theorem would then ensure us that $\bigcap_{\sigma\in\Sigma} \mathcal{W}_\sigma$ is dense, so that we could find an arbitrary small selfadjoint perturbation of P that makes the zeta and eta functions become singular at all points of the admissible set Σ.

In the light of this, one would like to have *explicit* constructions of such perturbations, not only in the category of pseudodifferential operators, but also by restricting ourselves to the smaller categories of differential operators and Dirac-type operators. The aim of this note is to exhibit those types of perturbations within the following classes of operators:

(i) First order selfadjoint elliptic differential operators (see Proposition 3.6).
(ii) Dirac-type operators (see Corollary 3.8).
(iii) Selfadjoint elliptic pseudo-differential operators (see Proposition 4.3 in the first-order case and Proposition 5.2 in the general case).

The proofs of those results are based on straightforward considerations on the noncommutative residue trace of Guillemin [Gu1] and Wodzicki [Wo2]. This trace provides us with a neat and unified treatment of the residues of the zeta and eta functions.

The result for the category (ii) of Dirac operators is an immediate by-product of the results for the category (i). In the odd-dimensional case this provides us with an alternative proof of a well-known result of Branson-Gilkey [BG, Theorem 4.3] about the genericity of the singularities of the eta function of a Dirac-type operator at positive odd integers $\leq n$ in even dimension and at all non-zero even integers $\leq n$ in odd dimension (those are the only points at which the eta function may have singularities; see below).

While Branson-Gilkey's result was obtained by making use of Riemannian invariant theory, and hence was specific to Dirac-type operators, our argument is purely analytic and applies to general first order selfadjoint elliptic differential operators. It would be interesting to obtain the genericity result of Branson-Gilkey for general first order selfadjoint elliptic differential operators in odd dimension by purely analytical means.

As it turns out, when P is a first order selfadjoint elliptic differential operator, the admissible sets at which the zeta and eta functions is much smaller than Σ (see Section 3). In particular, they are subsets of the set of positive integers $\leq n$. As a result, for such an operator, it is enough to use the simplest possible perturbations, namely, perturbations by real constants,

$$P + a, \qquad a \in \mathbb{R}.$$

In order to deal with first order ΨDOs the perturbations by constants are too crude, so we need to refine them. We consider perturbations of the form,

$$P_\epsilon := P + \epsilon|P|, \qquad P_{\epsilon,c} := P_\epsilon + cF(P)|P_\epsilon|^{-n}, \tag{1.2}$$

where $\epsilon \in (-1, 1)$ and $c \geq 0$ are small enough, and we have denoted by $|P|$ the absolute value of P and by $F(P) = P|P|^{-1}$ its sign operator.

The results when P is a ΨDO of order $m > 1$ follow from for the first order case by considering perturbations of the form,

$$P_{\epsilon,c,a} = F(Q_{\epsilon,c} + a)|Q_{\epsilon,c} + a|^m,$$

where a, c and ϵ are as above and $Q_{\epsilon,c}$ is defined as in (1.2) by substituting the first order operator $Q := F(P)|P|^{\frac{1}{m}}$ for P.

This note is organized as follows. In Section 2, we recall some background on the noncommutative residue trace and the zeta and eta functions of an elliptic operator. In Section 3, we prove the results for first order differential operators and Dirac-type operators. In Section 4, we deal with first-order pseudo-differential operators. In Section 5, we derive the results for higher order pseudo-differential operators.

In addition, two appendices are included. In Appendix A, for the reader's convenience, we recall the construction of the standard Fréchet space topologies on the spaces of ΨDOs of given orders. In Appendix B, we include corrections to Theorem 6.3 of [Po1], which misstates that if P is odd-class and its order has opposite parity to $\dim M$, then its eta function is regular at *all* integer points. As we shall explain, this property does hold true in odd dimension, but in even dimension we can only obtain regularity at *even* integer points (cf. Theorem B.1).

Acknowledgements. The authors wish to thank Peter Gilkey for fruitful discussions regarding the subject matter of this paper.

Notation. Throughout this paper we let M be a compact Riemannian manifold of dimension n and E be a Hermitian vector bundle over M of rank r. The Riemannian metric of M and the Hermitian metric of E then allow us to define an inner product on $L^2(M, E)$ with respect to which we define adjoints.

In addition, for $q \in \mathbb{C}$, we denote by $\Psi^q(M, E)$ the space of (classical) ΨDOs of order q acting on the sections of E. We shall endow $\Psi^q(M, E)$ with its standard Fréchet space topology, whose definition is recalled in Appendix A.

2. The noncommutative residue and the eta and zeta functions

In this section, we recall the basic definitions and properties of the noncommutative residue trace and the zeta and eta functions of an elliptic operator.

Define

$$\Psi^{\text{int}}(M, E) := \bigcup_{\Re q < -n} \Psi^q(M, E),$$

$$\Psi^{\mathbb{Z}}(M, E) := \bigcup_{q \in \mathbb{Z}} \Psi^q(M, E), \qquad \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M, E) := \bigcup_{q \in \mathbb{C}\setminus\mathbb{Z}} \Psi^q(M, E).$$

Notice that, as M is compact, the class $\Psi^{\mathbb{Z}}(M, E)$ is an algebra.

Any operator $P \in \Psi^{\text{int}}(M,E)$ is trace-class. Moreover, its Schwartz kernel $k_P(x,y)$ is continuous and the restriction to the diagonal $k_P(x,x)$ gives rise to a smooth $\operatorname{End} E$-valued density, i.e., a smooth section of the bundle $|\Lambda|(M) \otimes \operatorname{End} E$, where we denote by $|\Lambda|(M)$ the line bundle of densities over M. Therefore,

$$\operatorname{Trace}(P) = \int_M \operatorname{tr}_{E_x} k_P(x,x).$$

The map $P \to k_P(x,x)$ is holomorphic from $\Psi^{\text{int}}(M,E)$ to the Fréchet space $C^\infty(M, |\Lambda|(M) \otimes \operatorname{End} E)$, where holomorphy is meant with respect to the notion of holomorphic families of ΨDOs as defined in [Gu3] and [Ok]. Namely, if $(P(z))_{z\in\mathbb{C}}$ is any holomorphic family of ΨDOs with values in $\Psi^{\text{int}}(M,E)$, then $z \to k_P(x,x)$ is a holomorphic map from $\mathbb{C}$ to $C^\infty(M, |\Lambda|(M) \otimes \operatorname{End} E)$.

As observed by Guillemin [Gu2] and Wodzicki [Wo2] (see also [KV]), the map $P \to k_P(x,x)$ has a unique holomorphic extension as a map $P \to t_P(x)$ from $\Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,E)$ to $C^\infty(M, |\Lambda|(M) \otimes \operatorname{End} E)$. Therefore, the operator trace can be uniquely extended into the holomorphic functional on $\Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,E)$ defined by

$$\operatorname{TR} P := \int_M \operatorname{tr}_{E_x} t_P(x) \qquad \forall P \in \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,E).$$

In addition, the functional TR induces on the algebra $\Psi^{\mathbb{Z}}(M,E)$ the noncommutative residue trace of Guillemin [Gu1] and Wodzicki [Wo2] as follows.

Let $P \in \Psi^{\mathbb{Z}}(M,E)$ and, given some local coordinates and trivialization of E, let $p(x,\xi) \sim \sum_{j\geq 0} p_{m-j}(x,\xi)$ be the symbol of P with respect to these local coordinates and trivialization. Define

$$c_P(x) = (2\pi)^{-n} \int_{S^{n-1}} p_{-n}(x,\xi) d^{n-1}\xi, \tag{2.1}$$

where $p_{-n}(x,\xi)$ is the symbol of degree $-n$ of P and $d^{n-1}\xi$ denotes the surface measure of the sphere S^{n-1}.

At first glance, the function $c_P(x)$ depends on the choice of the local coordinates and trivialization. However, it can be shown that if we interpret it as a density, then it makes sense intrinsically on M as a smooth $\operatorname{End} E$-valued density, i.e., as an element of $C^\infty(M, |\Lambda|(M) \otimes \operatorname{End} E)$.

Furthermore, if $P \in \Psi^{\mathbb{Z}}(M,E)$ and $(P(z))_{z\in\mathbb{C}}$ is a holomorphic family of ΨDOs such that $P(0) = P$ and $\operatorname{ord} P(z) = z + \operatorname{ord} P$, then, near $z = 0$, the map $z \to t_{P(z)}(x)$ has at worst a simple pole singularity such that

$$\operatorname{Res}_{z=0} t_{P(z)}(x) = -c_P(x).$$

The *noncommutative residue* is the linear functional on $\Psi^{\mathbb{Z}}(M,E)$ defined by

$$\operatorname{Res} P := \int_M \operatorname{tr}_{E_x} c_P(x) \qquad \forall P \in \Psi^{\mathbb{Z}}(M,E). \tag{2.2}$$

Proposition 2.1 ([Gu3], [Wo2])**.** *The noncommutative residue has the following properties:*

(1) *It vanishes on $\Psi^{-(n+1)}(M,E)$ and on all differential operators. In particular, it vanishes on all smoothing operators.*

(2) *It is a trace on the algebra $\Psi^{\mathbb{Z}}(M,E)$, i.e.,*

$$\operatorname{Res}[P_1 P_2] = \operatorname{Res}[P_2 P_1] \qquad \forall P_j \in \Psi^{\mathbb{Z}}(M,E).$$

(3) *Let $P \in \Psi^{\mathbb{Z}}(M,E)$ and let $(P(z))_{z\in\mathbb{C}}$ be a holomorphic family of ΨDOs such that $P(0) = P$ and $\operatorname{ord} P(z) = z + \operatorname{ord} P$. Then, near $z = 0$, the map $z \to \operatorname{TR} P(z)$ has at worst a simple pole singularity such that*

$$\operatorname{Res}_{z=0} \operatorname{TR} P(z) = -\operatorname{Res} P.$$

In addition, if M is connected, then a result of Wodzicki [Wo1] (see also [LP], [Po3]) states that the noncommutative residue is the unique trace up to constant multiples on the algebra $\Psi^{\mathbb{Z}}(M,E)$.

Next, let $P : C^\infty(M,E) \to C^\infty(M,E)$ be a selfadjoint elliptic ΨDO of order m, $m \in \mathbb{N}$. Considered as an unbounded operator on $L^2(M,E)$ with domain the Sobolev space $L^2_m(M,E)$, the operator P is closed, selfadjoint and has a compact resolvent. Therefore, the spectral decomposition of P allows us to define a (Borel) functional calculus for P. In particular, for any $s \in \mathbb{C}$, this enables us to define the complex powers $P^s_\uparrow$ and $P^s_\downarrow$ by functional calculus associated to the functions,

$$\lambda^s_\uparrow := e^{is\arg_\uparrow s}, \qquad \lambda \in \mathbb{C} \setminus i[0,\infty),$$
$$\lambda^s_\downarrow := e^{is\arg_\downarrow s}, \qquad \lambda \in \mathbb{C} \setminus -i[0,\infty),$$

where $\arg_\uparrow$ (resp., $\arg_\downarrow$) is the continuous determination of the argument with values in $(-\frac{3\pi}{2}, \frac{\pi}{2})$ (resp., $(-\frac{\pi}{2}, \frac{3\pi}{2})$). In particular,

$$P^0_{\uparrow\downarrow} = 1 - \Pi_0(P), \qquad P^k_{\uparrow\downarrow} = P^k \quad \forall k \in \mathbb{Z} \setminus 0, \tag{2.3}$$

where $\Pi_0(P)$ is the orthogonal projection onto $\ker P$. It follows from the results of Seeley [Se] that $P^s_\uparrow$ and $P^s_\downarrow$ are ΨDOs of order ms and the families $(P^s_\uparrow)_{s\in\mathbb{C}}$ and $(P^s_\downarrow)_{s\in\mathbb{C}}$ are holomorphic families of ΨDOs (see also [Ok] concerning the holomorphy of these families).

The zeta functions of P are the meromorphic functions on $\mathbb{C}$ defined by

$$\zeta_\uparrow(P;s) := \mathrm{TR}\, P^{-s}_{\uparrow\downarrow} \qquad \text{and} \qquad \zeta_\downarrow(P;s) := \mathrm{TR}\, P^{-s}_{\uparrow\downarrow}.$$

In particular, for $\Re s > \frac{n}{m}$,

$$\zeta_{\uparrow\downarrow}(P;s) = \mathrm{Tr}\, P^{-s}_{\uparrow\downarrow} = \sum_{\lambda\in \mathrm{Sp}(P)\setminus 0} \lambda^{-s}_{\uparrow\downarrow},$$

where each non-zero eigenvalue of P is repeated according to multiplicity.

It follows from Proposition 2.1 that the functions $\zeta_{\uparrow\downarrow}(P;s)$ are analytic outside

$$\Sigma := \left\{ \frac{k}{m};\ k \in \mathbb{Z},\ k \le n \right\} \setminus \{0\}. \tag{2.4}$$

Moreover, near any $\sigma \in \Sigma$, there is at worst a simple pole singularity such that

$$m.\,\mathrm{Res}_{s=\sigma}\, \zeta_{\uparrow\downarrow}(P;s) = \mathrm{Res}\, P^{-\sigma}_{\uparrow\downarrow}. \tag{2.5}$$

Notice that at $s = 0$ we have

$$m.\,\mathrm{Res}_{s=0}\, \zeta_{\uparrow\downarrow}(P;s) = \mathrm{Res}\, P^0_{\uparrow\downarrow} = \mathrm{Res}\,[1 - \Pi_0(P)] = 0,$$

since the noncommutative residue vanishes on differential and smoothing operators.

When P is positive, i.e., its spectrum is contained in $[0,\infty)$, the zeta functions $\zeta_\uparrow(P;s)$ and $\zeta_\downarrow(P;s)$ agree, so in that case we shall drop the subscripts $\uparrow\downarrow$ and use the single notation $\zeta(P;s)$. For instance, the absolute value $|P| := \sqrt{P^2}$ is a positive selfadjoint elliptic ΨDO of order m. Thus, for this operator, we have a single zeta function,

$$\zeta(|P|;s) := \mathrm{TR}\, |P|^{-s}.$$

Notice that some authors refer to $\zeta(|P|;s)$ as the zeta function of P (see, e.g., [Gi2]).

Like the other zeta functions $\zeta_{\uparrow\downarrow}(P;s)$, the function $\zeta(|P|;s)$ is analytic on $\mathbb{C}\setminus\Sigma$ and it has at worst simple pole singularities on Σ such that

$$m.\,\mathrm{Res}_{s=\sigma}\, \zeta(|P|;s) = \mathrm{Res}\, |P|^{-\sigma} \qquad \forall \sigma \in \Sigma. \tag{2.6}$$

The eta function of a selfadjoint elliptic ΨDO was introduced by Atiyah-Patodi-Singer [APS1] as a tool to study the spectral asymmetry of such an operator. It can be defined as follows.

Let $F := P|P|^{-1}$ be the sign operator of P. This is a selfadjoint elliptic ΨDO of order zero such that

$$F^2 = 1 - \Pi_0(P). \tag{2.7}$$

The eta function of P is the meromorphic function on $\mathbb{C}$ defined by

$$\eta(P;s) := \mathrm{TR}\left[F|P|^{-s}\right].$$

In particular, for $\Re s > \frac{n}{m}$,

$$\eta(P;s) = \mathrm{Tr}\left[F|P|^{-s}\right] = \sum_{\lambda\in \mathrm{Sp}(P)\setminus 0} \mathrm{sign}(\lambda)|\lambda|^{-s},$$

where each non-zero eigenvalue of P is repeated according to multiplicity.

Like the zeta functions, the eta function is analytic on $\mathbb{C}\setminus\Sigma$ and has at worst simple pole singularities on Σ such that

$$m.\,\mathrm{Res}_{s=\sigma}\,\eta(P;s) = \mathrm{Res}\left[F|P|^{-\sigma}\right] \qquad \forall\sigma\in\Sigma. \tag{2.8}$$

Notice that at $s=0$ we have

$$m.\,\mathrm{Res}_{s=\sigma}\,\eta(P;s) = \mathrm{Res}\left[F|P|^{0}\right] = \mathrm{Res}\,F.$$

However, important results of Atiyah-Patodi-Singer [APS2] and Gilkey [Gi1] ensure us that $\mathrm{Res}\,F$ is zero, so that $\eta(P;s)$ is always regular at the origin.

3. First-order differential operators

In this section, we shall look at the singularities of the eta and zeta functions of first-order elliptic differential operators and Dirac-type operators. The main results of this section will be consequences of elementary lemmas, which we shall state for pseudodifferential operators since we will need to use of them in the next sections.

Let $P : C^\infty(M,E)\to C^\infty(M,E)$ be a selfadjoint elliptic first-order ΨDO and let us look at the effects of the simplest possible perturbations, namely, perturbations by constants,

$$P + a, \qquad a\in\mathbb{R}.$$

The operator $P+a$ is a selfadjoint elliptic first-order ΨDO with the same principal symbol as P.

In the sequel, given $Q\in\Psi^q(M,E)$, $q\in\mathbb{C}$, and $Q_j\in\Psi^{q-j}(M,E)$, $j=0,1,\cdots$, we shall write

$$Q \simeq \sum_{j\geq 0} Q_j$$

to mean that

$$Q - \sum_{j<N} Q_j \in \Psi^{q-N}(M,E) \qquad \forall N\in\mathbb{N}.$$

Lemma 3.1. *Let $a\in\mathbb{R}$. Then, for all $k\in\mathbb{N}$,*

$$(P+a)^{-k} \simeq \sum_{j\geq 0}\binom{-k}{j} a^j P^{-(k+j)}. \tag{3.1}$$

Proof. Let $\Pi_0(P)$ be the orthogonal projection onto $\ker P$. As $\Pi_0(P)$ is a smoothing operator, we have

$$P + a = P(1+aP^{-1}) + a\Pi_0(P) = P(1+aP^{-1}) \quad \mod \Psi^{-\infty}(M,E).$$

Let $N\in\mathbb{N}$ and set $u = aP^{-1}$. Then u is contained in $\Psi^{-1}(M,E)$ and

$$(1+u).\sum_{0\leq j<N}(-u)^j = 1-(-u)^N = 1 \quad \mod \Psi^{-N}(M,E).$$

Thus,

$$(P+a)\sum_{0\leq j<N}(-u)^j = P(1+u).\sum_{0\leq j<N}(-u)^j = P \mod \Psi^{1-N}(M,E).$$

Multiplying these equalities by $P^{-1}(P+a)^{-1}$ then gives

$$(P+a)^{-1} = P^{-1}.\sum_{0\leq j<N}(-u)^j \mod \Psi^{-1-N}(M,E). \tag{3.2}$$

Let $k\in\mathbb{N}$. From (3.2) we get

$$(P+a)^{-k} = P^{-k}.\Big(\sum_{0\leq j<N}(-u)^j\Big)^k \mod \Psi^{-k-N}(M,E). \tag{3.3}$$

Observe that, near $x=0$,

$$\Big(\sum_{0\leq j<N}(-x)^j\Big)^k = \big(1-(-x)^N\big)^k(1+x)^{-k} = \sum_{0\leq j<N}\binom{-k}{j}x^j + \mathrm{O}(x^N).$$

Since $\Big(\sum_{0\leq j<N}(-x)^j\Big)^k$ is a polynomial, we deduce that

$$\Big(\sum_{0\leq j<N}(-x)^j\Big)^k = \sum_{0\leq j<N}\binom{-k}{j}x^j \mod \mathrm{Span}\left\{x^N,\cdots x^{k(N-1)}\right\}.$$

It then follows that

$$\begin{aligned}\Big(\sum_{0\leq j<N}(-u)^j\Big)^k &= \sum_{0\leq j<N}\binom{-k}{j}u^j \mod \mathrm{Span}\left\{u^N,\cdots u^{k(N-1)}\right\}\\ &= \sum_{0\leq j<N}\binom{-k}{j}a^jP^{-j} \mod \Psi^{-N}(M,E).\end{aligned}$$

Combining this with (3.3) we get

$$(P+a)^{-k} = \sum_{0\leq j<N}\binom{-k}{j}a^jP^{-(k+j)} \mod \Psi^{-k-N}(M,E),$$

which proves (3.1). The proof is complete. □

Lemma 3.2. *The noncommutative residue* $\mathrm{Res}\,|P|^{-n}$ *is always* >0.

Proof. Let $p_1(x,\xi)$ be the principal symbol of P. Then $|P|^{-n}$ is a ΨDO of order $-n$ with principal symbol $|p_1(x,\xi)|^{-n}$. This symbol is positive-definite everywhere, so the density $c_{|P|^{-n}}(x)$ defined by (2.1) takes values in positive-definite sections of $\mathrm{End}\,E$, and hence the scalar density $\mathrm{tr}_E\,c_{|P|^{-n}}(x)$ is >0. Combining this with (2.2) shows that $\mathrm{Res}\,|P|^{-n}$ is >0, proving the lemma. □

In the sequel, we let $F := P|P|^{-1}$ be the sign of P. We observe that (2.7) implies that, for all $k\in\mathbb{Z}$,

$$F^k = \begin{cases} F & \text{if } k \text{ is odd,}\\ 1-\Pi_0(P) = 1 \bmod \Psi^{-\infty}(M,E) & \text{if } k \text{ is even.}\end{cases} \tag{3.4}$$

Lemma 3.3. *Let* $a\in\mathbb{R}$ *and denote by* F_a *the sign of* $P+a$. *Then*

$$F_a = F \mod \Psi^{-\infty}(M,E). \tag{3.5}$$

Proof. For $\lambda \in \mathbb{C}$, we shall denote by $\Pi_\lambda(P)$ the orthogonal projection onto $\ker(P-\lambda)$. If $\lambda \notin \operatorname{Sp} P$, then $\Pi_\lambda(P) = 0$, but if $\lambda \in \operatorname{Sp} P$, then $\Pi_\lambda(P)$ is a finite-rank projection and a smoothing operator. Moreover, with respect to the strong topology of $\mathcal{L}(L^2(M,E))$, we have

$$F = \sum_{\lambda>0} \Pi_\lambda(P) - \sum_{\lambda<0} \Pi_\lambda(P) \qquad \text{and} \qquad F_a = \sum_{\lambda>-a} \Pi_\lambda(P) - \sum_{\lambda<-a} \Pi_\lambda(P).$$

If $a > 0$, then

$$\begin{aligned} F_a &= \sum_{\lambda>0} \Pi_\lambda(P) + \sum_{0\geq\lambda>-a} \Pi_\lambda(P) - \Big(\sum_{\lambda<0} \Pi_\lambda(P) - \sum_{-a\leq\lambda<0} \Pi_\lambda(P)\Big) \\ &= F + 2\sum_{-a<\lambda<0} \Pi_\lambda(P) + \Pi_0(P) + \Pi_{-a}(P). \end{aligned} \tag{3.6}$$

Similarly, if $a < 0$, then

$$F_a = F - 2\sum_{0<\lambda<-a} \Pi_\lambda(P) - \Pi_0(P) - \Pi_{-a}(P).$$

In any case F_a and F agree up to a smoothing operator. The lemma is proved. □

In the remainder of the section we assume that P is a first order *differential operator*. As explained in the previous section, the zeta and eta functions of P may have singularities only at the points of the admissible set,

$$\Sigma := \Big\{k \in \mathbb{Z};\ k \leq n\Big\} \setminus \{0\}.$$

As we shall now see, the fact that P is a differential operator allows us to shrink the admissible sets at which the zeta and eta functions may have singularities.

Lemma 3.4. *If M has even dimension n, then*

(i) *$\zeta_\uparrow(P;s)$ and $\zeta_\downarrow(P;s)$ may have singularities only at positive integers $\leq n$.*
(ii) *$\zeta(|P|;s)$ may have singularities only at even positive integers $\leq n$.*
(iii) *$\eta(P;s)$ may have singularities only at odd positive integers $\leq n$.*

Proof. Let k be a non-zero integer $\leq n$. In view of (2.3) and (2.5) we have

$$\operatorname{Res}_{s=k} \zeta_{\uparrow\downarrow}(P;s) = \operatorname{Res} P_{\uparrow\downarrow}^{-k} = \operatorname{Res} P^{-k}. \tag{3.7}$$

Moreover, using (2.6) and (3.4) we get

$$\operatorname{Res}_{s=k} \zeta(|P|;s) = \operatorname{Res} |P|^{-k} = \operatorname{Res}\left[F^k P^{-k}\right] = \begin{cases} \operatorname{Res} P^{-k} & \text{if } k \text{ is even,} \\ \operatorname{Res}\left[FP^{-k}\right] & \text{if } k \text{ is odd.} \end{cases} \tag{3.8}$$

Likewise, combining (2.8) and (3.4) gives

$$\operatorname{Res}_{s=k} \eta(P;s) = \operatorname{Res}\left[F|P|^{-k}\right] = \operatorname{Res}\left[F^{k+1}P^{-k}\right] = \begin{cases} \operatorname{Res}\left[FP^{-k}\right] & \text{if } k \text{ is even,} \\ \operatorname{Res} P^{-k} & \text{if } k \text{ is odd.} \end{cases} \tag{3.9}$$

As the noncommutative residue of a differential operator is always zero, we see that $\operatorname{Res} P^{-k} = 0$ whenever k is negative. Combining this with (3.7) shows that $\zeta_{\uparrow\downarrow}(P;s)$ cannot have poles at negative integers, and hence may have poles only at positive integers $\leq n$. In addition, by using (3.8)–(3.9) we also see that $\zeta(|P|;s)$ (resp., $\eta(P;s)$) cannot have poles at even (resp., odd) negative integers.

Let $\Pi_-(P)$ be the orthogonal projection onto the negative eigenspace of P. As P is a differential operator of odd order and M has even dimension, it is proved on [Po1, page 1081] that

$$\operatorname{Res}\left[\Pi_-(P)P^{-k}\right] = \frac{1}{2}\operatorname{Res} P^{-k}.$$

As $FP^{-k} = (1 - 2\Pi_-(P))P^{-k}$ we see that

$$\operatorname{Res}\left[FP^{-k}\right] = \operatorname{Res}\left[(1 - 2\Pi_-(P))P^{-k}\right] = \operatorname{Res} P^{-k} - 2\operatorname{Res}\left[\Pi_-(P)P^{-k}\right] = 0.$$

Combining this with (3.8)–(3.9) shows that the function $\zeta(|P|; s)$ (resp., $\eta(P; s)$) cannot have poles at odd (resp., even) integers.

It follows from all this that the function $\zeta(|P|; s)$ (resp., $\eta(P; s)$) may have poles only at even (resp., odd) positive integers $\leq n$. This completes the proof. □

Lemma 3.5. *If M has odd dimension n, then*

(i) *$\zeta_\uparrow(P; s)$ and $\zeta_\downarrow(P; s)$ are entire functions.*

(ii) *$\zeta(|P|; s)$ may have singularities only at odd integers $\leq n$.*

(iii) *$\eta(P; s)$ may have singularities only at non-zero even integers $\leq n$.*

Proof. In the terminology of [KV], an operator $Q \in \Psi^m(M, E)$, $m \in \mathbb{Z}$, is said to be odd-class if, in any local coordinates and local trivialization of E, the symbol $q(x, \xi) \sim \sum_{j\geq 0} q_{m-j}(x, \xi)$ of Q satisfies

$$q_{m-j}(x, -\xi) = (-1)^{m-j} q_{m-j}(x, \xi) \qquad \forall j \in \mathbb{N}_0. \tag{3.10}$$

Using (2.1) it is not difficult to check that the above condition for $m - j = -n$ implies that the density $c_Q(x) = 0$ if n is odd (cf. [KV]). Thus the noncommutative residue of an odd-class ΨDO is always zero in odd dimension.

Any differential operator is odd-class and it not difficult to check that the parametrix of any elliptic odd-class ΨDO is again odd-class. Thus, for all $k \in \mathbb{Z}$, the operator P^{-k} is odd-class. As n is odd, we then deduce that

$$\operatorname{Res} P^{-k} = 0 \qquad \forall k \in \mathbb{Z}.$$

Combining this with (3.7)–(3.9) (and the regularity of $\eta(P; s)$ at $s = 0$) yields the lemma. □

Recaling that M has dimension n, we are now in a position to prove the main result of this section.

Proposition 3.6. *Suppose that P is a selfadjoint elliptic first-order differential operator. Then, for all but finitely many values of a,*

(1) *$\zeta_\uparrow(P + a; s)$ and $\zeta_\downarrow(P + a; s)$ are singular at all positive integers $\leq n$ if n is even.*

(2) *$\zeta(|P + a|; s)$ is singular at all positive even (resp., odd) integers $\leq n$ if n is even (resp., odd).*

(3) *$\eta(P + a; s)$ is singular at all positive odd (resp., even) integers $\leq n$ if n is even (resp., odd).*

Proof. Let $a \in \mathbb{R}$ and let k be a positive integer $\leq n$. In view of (3.7) we have

$$\operatorname{Res}_{s=k} \zeta_{\uparrow\downarrow}(P + a; s) = \operatorname{Res}(P + a)^{-k}. \tag{3.11}$$

Let $F_a := (P + a)|P + a|^{-1}$ be the sign operator of $P + a$. In addition, set $l = 0$ if n is even and $l = 1$ if n is odd. Then (3.8) shows that, if k and n have the same parity, then

$$\operatorname{Res}_{s=k} \zeta(|P + a|; s) = \operatorname{Res}\left[F_a^l (P + a)^{-k}\right]. \tag{3.12}$$

Likewise, using (3.9) we see that, if k and n have opposite parities, then

$$\operatorname{Res}_{s=k} \eta(P + a; s) = \operatorname{Res}\left[F_a^l (P + a)^{-k}\right]. \tag{3.13}$$

It follows from (3.11)–(3.13) that in order to prove the proposition it is enough to show, that for any positive integer $k \leq n$ and for all but finitely values of a,

$$\operatorname{Res}\left[F_a^l (P + a)^{-k}\right] \neq 0. \tag{3.14}$$

Thanks to Lemma 3.1 we know that

$$(P+a)^{-k} = \sum_{0\leq j\leq n-k} \binom{-k}{j} a^j P^{-(k+j)} \qquad \mod \Psi^{-(n+1)}(M,E),$$

Combining this with Lemma 3.3 we get

$$\begin{aligned} F_a^l(P+a)^{-k} &= F^l(P+a)^{-k} \quad \mod \Psi^{-\infty}(M,E), \\ &= \sum_{0\leq j\leq n-k} \binom{-k}{j} a^j F^l P^{-(k+j)} \quad \mod \Psi^{-(n+1)}(M,E). \end{aligned}$$

As the noncommutative residue vanishes on $\Psi^{-(n+1)}(M,E)$, we deduce that

$$\operatorname{Res}\left[F_a^l(P+a)^{-k}\right] = \sum_{0\leq j\leq n-k} \binom{-k}{j} a^j \operatorname{Res}\left[F^l P^{-(k+j)}\right]. \tag{3.15}$$

Thus $\operatorname{Res}\left[F_a^l(P+a)^{-k}\right]$ is polynomial in a of degree $n-k$ whose leading coefficient is a non-zero multiple of $\operatorname{Res}\left[F^l P^{-n}\right]$.

If n is even, then $F^l P^{-n} = P^{-n} = |P|^{-n}$. If n is odd, then (3.4) implies that $F^l P^{-n} = F^n P^{-n} = |P|^{-n}$. Combining this with Lemma 3.2 shows that, in both cases,

$$\operatorname{Res}\left[F^l P^{-n}\right] = \operatorname{Res}|P|^{-n} > 0.$$

It then follows that $\operatorname{Res}\left[F_a^l(P+a)^{-k}\right]$ is a *non-zero* polynomial, and so it may vanish for at most finitely many values of a. This completes the proof. □

Corollary 3.7. *For a generic selfadjoint elliptic first-order differential operator $P : C^\infty(M,E) \to C^\infty(M,E)$ the following hold:*

(i) *$\zeta_\uparrow(P;s)$ and $\zeta_\downarrow(P;s)$ are singular at all positive integers $\leq n$ if n is even.*
(ii) *$\zeta(|P|;s)$ is singular at all positive even (resp., odd) integers $\leq n$ if n is even (resp., odd).*
(iii) *$\eta(P;s)$ is singular at all positive odd (resp., even) integers $\leq n$ if n is even (resp., odd).*

Finally, recall that a Dirac-type operator is a selfadjoint first order differential operator $D : C^\infty(M,E) \to C^\infty(M,E)$ such that the principal symbol of D^2 satisfies

$$\sigma_2(D^2)(x,\xi) = |\xi|^2 . \mathrm{id}_{E_x} \qquad \forall (x,\xi) \in T^*M,$$

where $|\xi|^2 := g^{ij}(x)\xi_i\xi_j$ is the Riemannian metric of T^*M.

By a result of Branson-Gilkey [BG, Theorem 4.3], if we restrict ourselves to the class of Dirac-type operators then, *generically*, the eta function has singularities at all positive odd integers $k < n$ if n is even and at all non-zero even integers $< n$ if n if odd.

The proof of Branson-Gilkey's result relied on the Riemannian invariant theory of [ABP] and [Gi2]. While the use of the Riemannian invariant theory is an extremely powerfool tool to get precised information on the coefficients of the heat kernel asymptotics, it is rather specific to Laplace-type and Dirac-type operators. As a result, the arguments of [BG] do not extend to general differential operators. Therefore, it would be desirable to have a purely analytical proof of Branson-Gilkey's result.

Observe that the class of Dirac-type operator is invariant under perturbations by constants. Therefore, specializing Proposition 3.6 to Dirac-type operators we immediately get

Corollary 3.8. *Let $D : C^\infty(M,E) \to C^\infty(M,E)$ be a Dirac-type operator. Then we always can perturbate D by an arbitrary small real constant a so that*

(i) $\zeta_{\uparrow}(D;s)$ *and* $\zeta_{\downarrow}(D;s)$ *are singular at all positive integers* $\leq n$ *if* n *is even.*

(ii) $\zeta(|D|;s)$ *is singular at all positive even (resp., odd) integers* $\leq n$ *if* n *is even (resp., odd).*

(iii) $\eta(D;s)$ *is singular at all positive odd (resp., even) integers* $\leq n$ *if* n *is even (resp., odd).*

In particular, in the even-dimensional case this provides us with a purely analytical proof of the aforementioned genericity result of Branson and Gilkey.

4. First-order pseudodifferential operators

In this section, we let $P : C^{\infty}(M,E) \to C^{\infty}(M,E)$ be a selfadjoint elliptic first-order ΨDO. Then the set of admissible points at which the eta function $\eta(P;s)$ and the zeta functions $\zeta_{\uparrow\downarrow}(P;s)$ and $\zeta(|P|;s)$ are allowed to have singularities is

$$\Sigma = \{k \in \mathbb{Z};\ k \leq n\} \setminus \{0\}\,. \tag{4.1}$$

That is, Σ consists of all non-zero integers $\leq n$.

In the sequel, we let $F := P|P|^{-1}$ the sign operator of P.

Lemma 4.1. *If the following three conditions simultaneously hold*

$$\operatorname{Res}\left[F|P|^{-n}\right] \neq 0, \qquad \operatorname{Res} P \neq 0, \qquad \operatorname{Res}|P| \neq 0, \tag{4.2}$$

then, for all but countably many values of the real number a*, the function* $\eta(P+a;s)$ *and the zeta functions* $\zeta_{\uparrow\downarrow}(P+a;s)$ *and* $\zeta(|P+a|;s)$ *are singular at all non-zero integers* $\leq n$*.*

Proof. Let $a \in \mathbb{R}$ and denote by $F_a = (P+a)|P+a|^{-1}$ the sign operator of $P+a$. In addition, let k be a non-zero integer $\leq n$. The formulas (3.7)–(3.9) remain valid for $P+a$. Therefore, each of the residues at $s=k$ of the zeta functions $\zeta_{\uparrow\downarrow}(P+a;s)$ and $\zeta(|P+a|;s)$ and the eta function $\eta(P;s)$ is equal to one of the following

$$\operatorname{Res}(P+a)^{-k} \qquad \text{or} \qquad \operatorname{Res}\left[F_a(P+a)^{-k}\right].$$

Furthermore, as by Lemma 3.3 the operators F and F_a agree up to a smoothing operator and the noncommutative residue vanishes on smoothing operators, we get

$$\operatorname{Res}\left[F_a(P+a)^{-k}\right] = \operatorname{Res}\left[F(P+a)^{-k}\right].$$

Thus, in order to prove the lemma it is enough to show that if the conditions (4.2) hold, then, for each non-zero integer $k \leq n$, all but finitely values of a satisfy

$$\operatorname{Res}(P+a)^{-k} \neq 0 \qquad \text{and} \qquad \operatorname{Res}\left[F(P+a)^{-k}\right] \neq 0. \tag{4.3}$$

Let $l \in \{0,1\}$. If k is a positive integer $\leq n$, then, as in (3.15), we have

$$\operatorname{Res}\left[F^l(P+a)^{-k}\right] = \sum_{0\leq j\leq n-k} \binom{-k}{j} a^j \operatorname{Res}\left[F^l P^{-(k+j)}\right].$$

Combining this with (3.4) we see that $\operatorname{Res}\left[F^l(P+a)^{-k}\right]$ is a polynomial in a whose leading coefficient is a non-zero multiple of

$$\operatorname{Res}\left[F^l P^{-n}\right] = \operatorname{Res}\left[F^{l+n}|P|^{-n}\right] = \begin{cases} \operatorname{Res}|P|^{-n} & \text{if } n = l \bmod 2, \\ \operatorname{Res}\left[F|P|^{-n}\right] & \text{if } n = l+1 \bmod 2. \end{cases}$$

If k is a negative integer, then the binomial formula gives

$$F^l(P+a)^{-k} = F^l(P+a)^{|k|} = \sum_{0\leq j\leq |k|} \binom{|k|}{j} a^{|k|-j} F^l P^j,$$

Combining this with the fact that $\operatorname{Res}[1] = \operatorname{Res} F = 0$ (cf. the end of Section 2), we obtain

$$\operatorname{Res}\left[F^l(P+a)^{-k}\right] = \sum_{0\leq j\leq |k|} \binom{|k|}{j} a^{|k|-j} \operatorname{Res}\left[F^l P^j\right] = \sum_{1\leq j\leq |k|} \binom{|k|}{j} a^{|k|-j} \operatorname{Res}\left[F P^j\right].$$

This shows that $\operatorname{Res}\left[F^l(P+a)^{-k}\right]$ is a polynomial in a of degree $|k|-1$ whose leading coefficient is equal to

$$\operatorname{Res}\left[F^l P\right] = \begin{cases} \operatorname{Res} P & \text{if } l = 0, \\ \operatorname{Res}[FP] = \operatorname{Res}|P| & \text{if } l = 1. \end{cases}$$

It follows from all this that, for any non-zero integer $k \leq n$, both $\operatorname{Res}(P+a)^{-k}$ and $\operatorname{Res}\left[F(P+a)^{-k}\right]$ are polynomials in a whose leading coefficients are non-zero multiples of one of the following noncommutative residues:

$$\operatorname{Res}|P|^{-n}, \qquad \operatorname{Res}\left[F|P|^{-n}\right], \qquad \operatorname{Res} P, \qquad \operatorname{Res}|P|.$$

Thanks to Lemma 3.2 we know that the first of these noncommutative residues is always non-zero. Therefore, if the other three are non-zero, then $\operatorname{Res}(P+a)^{-k}$ and $\operatorname{Res}\left[F(P+a)^{-k}\right]$ both are *non-zero* polynomials in a, and hence vanish for at most finitely many values of a. As mentioned above, this proves the lemma. □

We shall now construct perturbations of P that ensure us that the three conditions in (4.2) hold. To this end, for $-1 < \epsilon < 1$ and $c \geq 0$, we define

$$P_\epsilon := P + \epsilon|P|, \qquad P_{\epsilon,c} := P_\epsilon + cF|P_\epsilon|^{-n}. \tag{4.4}$$

The operators P_ϵ and $P_{\epsilon,c}$ are selfadjoint elliptic first-order ΨDOs with the same principal symbols.

Lemma 4.2. *If ϵ is small enough, then, for all but finitely many values of c, the operator $P_{\epsilon,c}$ satisfy the three conditions (4.2).*

Proof. In the sequel, we denote by $\Pi_0(P)$ the orthogonal projection onto $\ker P$ and by $\Pi_+(P)$ (resp., $\Pi_-(P)$) the orthogonal projection onto the positive (resp., negative) eigenspaces of P. Thus,

$$\Pi_+(P) + \Pi_-(P) = 1 - \Pi_0(P), \qquad F = \Pi_+(P) - \Pi_-(P), \tag{4.5}$$

$$\Pi_\pm(P) = \frac{1}{2}(1 \pm F) - \frac{1}{2}\Pi_0(P). \tag{4.6}$$

In particular, Eq. (4.6) shows that $\Pi_+(P)$ and $\Pi_-(P)$ are zeroth order ΨDOs.

We shall use similar notations for the corresponding projections associated to P_ϵ and $P_{\epsilon,c}$. In addition, we let $F_\epsilon := P_\epsilon|P_\epsilon|^{-1}$ and $F_{\epsilon,c} := P_{\epsilon,c}|P_{\epsilon,c}|^{-1}$ be the respective sign operators of P_ϵ and $P_{\epsilon,c}$.

By definition,

$$P_\epsilon = P + \epsilon|P| = (1+\epsilon F)P.$$

Thus P and P_ϵ have same null space, and if $\lambda \in \operatorname{Sp} P \setminus \{0\}$, then $(1+\epsilon\operatorname{sign}(\lambda))\lambda$ is an eigenvalue of P_ϵ with same eigenspace and sign as λ. Therefore,

$$\Pi_0(P_\epsilon) = \Pi_0(P), \qquad \Pi_\pm(P_\epsilon) = \Pi_\pm(P), \qquad F_\epsilon = F. \tag{4.7}$$

It then follows that

$$|P_\epsilon| = F_\epsilon P_\epsilon = F(1+\epsilon F)P = (1+\epsilon F)|P|, \tag{4.8}$$

$$F|P_\epsilon|^{-n} = F(1+\epsilon F)^{-n}|P|^{-n}. \tag{4.9}$$

Let $\lambda \in \operatorname{Sp} P$. If $\lambda > 0$ (resp., $\lambda < 0$), then $(1+\epsilon)^{-n}$ (resp., $(1-\epsilon)^{-n}$) is an eigenvalue of $(1+\epsilon F)^{-n}$ with eigenspace $\ker(P-\lambda)$. If $\lambda = 0$, then 1 is an eigenvalue of $(1+\epsilon F)^{-n}$ with eigenspace $\ker P$. It follows that

$$(1+\epsilon F)^{-n} = (1+\epsilon)^{-n}\Pi_+(P) + (1-\epsilon)^{-n}\Pi_-(P) + \Pi_0(P).$$

Let $u(\epsilon)$ and $v(\epsilon)$ be the functions on $(-1,1)$ defined by

$$u(\epsilon)=\frac{1}{2}\left((1+\epsilon)^{-n}+(1-\epsilon)^{-n}\right)\quad\text{and}\quad v(\epsilon)=\frac{1}{2}\left((1+\epsilon)^{-n}-(1-\epsilon)^{-n}\right),$$

so that $(1\pm\epsilon)^{-n}=u(\epsilon)\pm v(\epsilon)$. Then $(1+\epsilon)^{-n}\Pi_+(P)+(1-\epsilon)^{-n}\Pi_-(P)$ is equal to

$$\begin{aligned}(4.10)\quad &(u(\epsilon)+v(\epsilon))\,\Pi_+(P)+(u(\epsilon)-v(\epsilon))\,\Pi_-(P)\\ &=u(\epsilon)\,(\Pi_+(P)+\Pi_-(P))+v(\epsilon)\,(\Pi_+(P)-\Pi_-(P))\\ &=u(\epsilon)\,(1-\Pi_0(P))+v(\epsilon)F.\end{aligned}$$

Combining (4.9) and (4.10) we get

$$(4.11)\quad F|P_\epsilon|^{-n}=F\,(u(\epsilon)\,(1-\Pi_0(P))+v(\epsilon)F)\,|P|^{-n}=u(\epsilon)F|P|^{-n}+v(\epsilon)|P|^{-n}.$$

Therefore, as $\epsilon\to 0$, we have

$$\begin{aligned}\operatorname{Res}\left[F|P_\epsilon|^{-n}\right]&=u(\epsilon)\operatorname{Res}\left[F|P|^{-n}\right]+v(\epsilon)\operatorname{Res}|P|^{-n}\\ &=\left(1+\mathrm{O}(\epsilon^2)\right)\operatorname{Res}\left[F|P|^{-n}\right]+\left(-n\epsilon+\mathrm{O}(\epsilon^3)\right)\operatorname{Res}|P|^{-n}\\ &=\operatorname{Res}\left[F|P|^{-n}\right]-n\epsilon\operatorname{Res}|P|^{-n}+\mathrm{O}(\epsilon^2).\end{aligned}$$

By Lemma 3.2 the noncommutative residue $\operatorname{Res}|P|^{-n}$ is always >0, so we see that if ϵ is small enough, then $\operatorname{Res}F|P_\epsilon|^{-n}$ is non-zero.

From now on we choose ϵ so that $\operatorname{Res}F|P_\epsilon|^{-n}\neq 0$. In view of (4.4) we have

$$\operatorname{Res}P_{\epsilon,c}=\operatorname{Res}P_\epsilon+c\operatorname{Res}F|P_\epsilon|^{-n},$$

so we see that for all values of c, except maybe one, the noncommutative residue $\operatorname{Res}P_{\epsilon,c}$ is non-zero.

The operators P_ϵ and $P_{\epsilon,c}=P_\epsilon+cF|P_\epsilon|^{-n}$ have the same null space. If λ is a non-zero eigenvalue of P_ϵ, then $\lambda+c\operatorname{sign}(\lambda)|\lambda|^{-n}$ is an eigenvalue of $P_{\epsilon,c}$ with eigenspace $\ker(P_\epsilon-\lambda)$ and the same sign as λ. Combining this with (4.7) shows that

$$(4.12)\qquad \Pi_0(P_{\epsilon,c})=\Pi_0(P_\epsilon)=\Pi_0(P)\qquad\text{and}\qquad F_{\epsilon,c}=F_\epsilon=F.$$

Therefore,

$$|P_{\epsilon,c}|=F_{\epsilon,c}P_{\epsilon,c}=F_\epsilon\left(P_\epsilon+cF_\epsilon|P_\epsilon|^{-n}\right)=|P_\epsilon|+c|P_\epsilon|^{-n},$$

where we used (2.7), and hence

$$\operatorname{Res}|P_{\epsilon,c}|=\operatorname{Res}|P_\epsilon|+c\operatorname{Res}|P_\epsilon|^{-n}.$$

Since $\operatorname{Res}|P_\epsilon|^{-n}\neq 0$, we see that for all, except maybe one value of c, the noncommutative residue $\operatorname{Res}|P_{\epsilon,c}|$ is non-zero.

Since P_ϵ and $P_{\epsilon,c}$ have the same principal symbols, the principal symbols of $|P_\epsilon|^{-n}$ and $|P_{\epsilon,c}|^{-n}$ agree, i.e., those operators differ by a ΨDO of order $\leq -(n+1)$. As $F_{\epsilon,c}=F$, we see that $F_{\epsilon,c}|P_{\epsilon,c}|^{-n}$ and $F|P_\epsilon|^{-n}$ agree modulo an element in $\Psi^{-(n+1)}(M,E)$, and hence $\operatorname{Res}F_{\epsilon,c}|P_{\epsilon,c}|^{-n}=\operatorname{Res}F|P_\epsilon|^{-n}\neq 0$.

It follows from all this that if ϵ is small enough, then, for all but finitely many values of c, all three conditions in (4.2) are satisfied by $P_{\epsilon,c}$. The lemma is proved. □

Combining Lemma 4.1 and Lemma 4.2 we immediately get

Proposition 4.3. *If ϵ is small enough, then, for all but countably many values of a and all but finitely many values of c, the eta function $\eta(P_{\epsilon,c}+a;s)$ and the zeta functions $\zeta_{\uparrow\downarrow}(P_{\epsilon,c}+a;s)$ and $\zeta(|P_{\epsilon,c}+a|;s)$ are singular at all the non-zero integers $\leq n$, that is, at all points of the admissible set (4.1).*

Observe that from (4.4) and (4.11) we get

$$P_{\epsilon,c} + a = P + T_{\epsilon,c,a},$$
$$T_{\epsilon,c,a} := \epsilon|P| + cF|P_\epsilon|^{-n} + a = \epsilon|P| + cu(\epsilon)F|P|^{-n} + cv(\epsilon)|P|^{-n} + a.$$

Clearly, as $(\epsilon, c, a) \to (0,0,0)$ the operator $T_{\epsilon,c,a}$ converges to 0 in $\Psi^1(M,E)$. Therefore, we obtain

Corollary 4.4. *We always can perturbate P by an arbitrary small selfadjoint element of $\Psi^1(M,E)$ in such way that the eta function $\eta(P;s)$ and the zeta functions $\zeta_{\uparrow\downarrow}(P;s)$ and $\zeta(|P|;s)$ become singular at all points of the admissible set (4.1).*

5. General Case

In this section, we extend the results of the previous section to operators of higher order. In the sequel, if $P : C^\infty(M,E) \to C^\infty(M,E)$ is any given selfadjoint elliptic ΨDO, then we shall denote by $F(P)$ its sign.

Throughout this section we let $P : C^\infty(M,E) \to C^\infty(M,E)$ be a selfadjoint order elliptic ΨDO of order m, $m \in \mathbb{N}$. Then the set of admissible points at which the eta and zeta functions of P are allowed to have singularities is

$$\Sigma := \left\{ \frac{k}{m};\ k \in \mathbb{Z},\ k \leq n \right\} \setminus \{0\}. \tag{5.1}$$

Let $Q : C^\infty(M,E) \to C^\infty(M,E)$ be the operator defined by

$$Q := F(P)|P|^{\frac{1}{m}}.$$

Then Q is a selfadjoint elliptic first-order ΨDO and we have

$$F(Q) = F(P), \qquad |P| = |Q|^m, \qquad P = F(Q)|Q|^m. \tag{5.2}$$

Lemma 5.1. *Let k be a non-zero integer $\leq n$. Then the following are equivalent:*

(i) *The eta function $\eta(P;s)$ and the zeta functions $\zeta_{\uparrow\downarrow}(P;s)$ and $\zeta(|P|;s)$ are singular at $s = \frac{k}{m}$.*
(ii) *The eta function $\eta(Q;s)$ and the zeta functions $\zeta_{\uparrow\downarrow}(Q;s)$ and $\zeta(|Q|;s)$ are singular at $s = k$.*
(iii) *Both $\operatorname{Res}|P|^{-\frac{k}{m}}$ and $\operatorname{Res}\left[F(P)|P|^{-\frac{k}{m}}\right]$ are non-zero.*

Proof. Set $F = F(P)$. As $F = F(Q)$ and $|P|^{\frac{k}{m}} = |Q|^k = F(Q)^k Q^k$, by (3.4) we see that (iii) is equivalent to having

$$\operatorname{Res} Q^{-k} \neq 0 \qquad \text{and} \qquad \operatorname{Res}\left[F(Q)Q^{-k}\right] \neq 0.$$

These conditions are exactly the conditions in (4.3) for Q in the special case $a = 0$. Thus, the equivalence between (ii) and (iii) follows from the beginning of the proof of Lemma 4.1.

It remains to prove that (i) and (iii) are equivalent. Using (2.6) and (2.8) we get

$$m \operatorname{Res}_{s=\frac{k}{m}} \zeta(|P|;s) = \operatorname{Res}|P|^{-\frac{k}{m}}, \quad m \operatorname{Res}_{s=\frac{k}{m}} \eta(P;s) = \operatorname{Res}\left[F|P|^{-\frac{k}{m}}\right].$$

Therefore $\eta(P;s)$ and $\zeta(|P|;s)$ are singular at $s = \frac{k}{m}$ if and only if (iii) holds.

To complete the proof it enough to show that if (iii) holds, then the zeta functions $\zeta_\uparrow(P;s)$ and $\zeta_\downarrow(P;s)$ too are singular at $s = \frac{k}{m}$.

Set $\sigma = \frac{k}{m}$. By the very definition of the powers $P_{\uparrow\downarrow}^{-\sigma}$ (cf. Section 2), we have

$$P_\uparrow^{-\sigma} = \Pi_+(P)|P|^{-\sigma} + e^{i\pi\sigma}\Pi_-(P)|P|^{-\sigma},$$
$$P_\downarrow^{-\sigma} = \Pi_+(P)|P|^{-\sigma} + e^{-i\pi\sigma}\Pi_-(P)|P|^{-\sigma},$$

where $\Pi_+(P)$ (resp., $\Pi_-(P)$) is the orthogonal projection onto the positive (resp., negative) eigenspaces of P. Combining this with (4.6) we get

$$\begin{aligned} P_\uparrow^{-\sigma} &= \left\{\frac{1}{2}(1+F)-\Pi_0(P)\right\}|P|^{-\sigma}+e^{i\pi\sigma}\left\{\frac{1}{2}(1-F)-\Pi_0(P)\right\}|P|^{-\sigma} \\ &= \frac{1}{2}(1+e^{i\pi\sigma})|P|^{-\sigma}+\frac{1}{2}(1-e^{i\pi\sigma})F|P|^{-\sigma}. \end{aligned}$$

As $\frac{1}{2}(1\pm e^{i\pi\sigma}) = e^{i\pi\frac{\sigma}{2}}\frac{1}{2}(e^{-i\pi\frac{\sigma}{2}}\pm e^{i\pi\frac{\sigma}{2}})$, we obtain

$$P_\uparrow^{-\sigma} = e^{i\pi\frac{\sigma}{2}}\left\{\cos\left(\frac{\pi\sigma}{2}\right)|P|^{-\sigma}-i\sin\left(\frac{\pi\sigma}{2}\right)F|P|^{-\sigma}\right\}.$$

Combining this with (2.5) then gives

$$\begin{aligned} m.\operatorname{Res}_{s=\sigma}\zeta_\uparrow(P;s) &= \operatorname{Res}P_\uparrow^{-\sigma} \\ &= e^{i\pi\frac{\sigma}{2}}\left\{\cos\left(\frac{\pi\sigma}{2}\right)\operatorname{Res}\left[|P|^{-\sigma}\right]-i\sin\left(\frac{\pi\sigma}{2}\right)\operatorname{Res}\left[F|P|^{-\sigma}\right]\right\}. \end{aligned} \tag{5.3}$$

We claim that $\operatorname{Res}|P|^{-\sigma}$ and $\operatorname{Res}[F|P|^{-\sigma}]$ are real numbers. Indeed, as for $\Re s > \frac{n}{m}$ we have $\operatorname{Trace}[|P|^{-s}] = \overline{\operatorname{Trace}[(|P|^{-s})^*]} = \overline{\operatorname{Trace}[|P|^{-\bar{s}}]}$, the meromorphic functions $\operatorname{TR}|P|^{-s}$ and $\overline{\operatorname{TR}|P|^{-\bar{s}}}$ agree. Combining this with (2.6) then shows that

$$\operatorname{Res}|P|^{-\sigma} = m.\operatorname{Res}_{s=\sigma}\operatorname{TR}|P|^{-s} = m.\operatorname{Res}_{s=\sigma}\overline{\operatorname{TR}P|^{-\bar{s}}} = \overline{\operatorname{Res}|P|^{-\sigma}},$$

proving that $\operatorname{Res}|P|^{-\sigma}$ is a real number. A similar argument shows that $\operatorname{Res}[F|P|^{-\sigma}]$ too is a real number.

Since $\operatorname{Res}|P|^{-\sigma}$ and $\operatorname{Res}[F|P|^{-\sigma}]$ are real numbers, Eq. (5.3) shows that the real and imaginary parts of $e^{-i\pi\frac{\sigma}{2}}\operatorname{Res}_{s=\sigma}\zeta_\uparrow(P;s)$ are given by

$$\begin{aligned} m.\Re\left(e^{-i\pi\frac{\sigma}{2}}\operatorname{Res}_{s=\sigma}\zeta_\uparrow(P;s)\right) &= \cos\left(\frac{\pi\sigma}{2}\right)\operatorname{Res}|P|^{-\sigma}, \\ m.\Im\left(e^{-i\pi\frac{\sigma}{2}}\operatorname{Res}_{s=\sigma}\zeta_\uparrow(P;s)\right) &= -\sin(\frac{\pi\sigma}{2})\operatorname{Res}\left[F|P|^{-\sigma}\right]. \end{aligned}$$

Since $\cos(\frac{\pi\sigma}{2})$ and $\sin(\frac{\pi\sigma}{2})$ cannot be simultaneously zero, we deduce that if both $\operatorname{Res}|P|^{-\sigma}$ and $\operatorname{Res}[F|P|^{-\sigma}]$ are non-zero, then $\operatorname{Res}_{s=\sigma}\zeta_\uparrow(P;s)$ must be non-zero. This means that if (iii) holds, then $\zeta_\uparrow(P;s)$ is singular at $s=\frac{k}{m}$.

A similar argument shows that if (iii) holds, then the zeta function $\zeta_\downarrow(P;s)$ too is singular at $s=\frac{k}{m}$. This completes the proof. □

For $(\epsilon,c)\in(-1,1)\times[0,\infty)$ define Q_ϵ and $Q_{\epsilon,c}$ as in (4.4), i.e.,

$$Q_\epsilon := Q+\epsilon|Q|, \qquad Q_{\epsilon,c} := Q_\epsilon + cF(Q)|Q_\epsilon|^{-n}.$$

Let $a\in\mathbb{R}$. The operator $Q_{\epsilon,c}+a$ is a selfadjoint elliptic first-order ΨDO, so we define a selfadjoint elliptic ΨDO of order m by letting

$$P_{\epsilon,c,a} := F(Q_{\epsilon,c}+a)|Q_{\epsilon,c}+a|^m.$$

Proposition 5.2. (1) *We can make $P_{\epsilon,c,a}$ be arbitrary close to P by picking up any sufficiently small data (ϵ,c,a).*

(2) *If ϵ is small enough, then, for all but countably many values of a and all but finitely many values of c, the eta function $\eta(P_{\epsilon,c,a};s)$ and the zeta functions $\zeta_{\uparrow\downarrow}(P_{\epsilon,c,a};s)$ and $\zeta(|P_{\epsilon,c,a}|;s)$ are singular at all points of the admissible set (5.1).*

Proof. As $F(P_{\epsilon,c,a}) = F(Q_{\epsilon,c}+a)$ and $|P_{\epsilon,c,a}| = |Q_{\epsilon,c}+a|^m$, we have

$$Q_{\epsilon,c}+a = F(Q_{\epsilon,c}+a)|Q_{\epsilon,c}+a| = F(P_{\epsilon,c,a})|P_{\epsilon,c,a}|^{\frac{1}{m}}.$$

Therefore, Lemma 5.1 shows that, for any non-zero integer $k\le n$, the following are equivalent:

(i) The eta function $\eta(P_{\epsilon,c,a};s)$ and the zeta functions $\zeta_{\uparrow\downarrow}(P_{\epsilon,c,a};s)$ and $\zeta(|P_{\epsilon,c,a}|;s)$ are singular at $s=\frac{k}{m}$.

(ii) The eta function $\eta(Q_{\epsilon,c}+a;s)$ and the zeta functions $\zeta_{\uparrow\downarrow}(Q_{\epsilon,c}+a;s)$ and $\zeta(|Q_{\epsilon,c}+a|;s)$ are singular at $s=k$.

Moreover, Proposition 4.3 asserts that, if ϵ is small enough, then, for all but countably many values of a and all but finitely many values of c, the eta function $\eta(Q_{\epsilon,c}+a;s)$ and the zeta functions $\zeta_{\uparrow\downarrow}(Q_{\epsilon,c}+a;s)$ and $\zeta(|Q_{\epsilon,c}+a|;s)$ are singular at all non-zero integers $\leq n$. Combining this with the equivalence of (i) and (ii) yields the 2nd part of the proposition.

In order to prove the 1st part of the proposition we have to show that $P_{\epsilon,c,a}$ converges to P as (ϵ,c,a) goes to $(0,0,0)$. The only (minor) difficulty comes from the term $F(Q_{\epsilon,c}+a)$ which does not depend continuously on the data (ϵ,c,a).

Let $f_{\epsilon,c}:\mathbb{R}\to\mathbb{R}$ be the function defined by

$$f_{\epsilon,c}(\lambda)=\lambda+\epsilon|\lambda|+c\,\mathrm{sign}(\lambda)\,|\lambda+\epsilon|\lambda||^{-n}\quad\text{if }\lambda\neq 0\ ,\qquad f_{\epsilon,c}(0)=0.$$

Then $Q_{\epsilon,c}=f_{\epsilon,c}(Q)$, and hence

$$\mathrm{Sp}\,Q_{\epsilon,c}=f_{\epsilon,c}\left(\mathrm{Sp}\,Q\right).$$

Set $\mu:=\inf\{|\lambda|;\ \lambda\in\mathrm{Sp}\,Q,\lambda\neq 0\}=\|Q^{-1}\|^{-1}$. If $\lambda\geq\mu$, then

$$f_{\epsilon,c}(\lambda)=\lambda+\epsilon\lambda+c\lambda^{-n}\geq(1+\epsilon)\lambda\geq(1+\epsilon)\mu.$$

Likewise, if $\lambda\leq-\mu$, then

$$f_{\epsilon,c}(\lambda)=\lambda-\epsilon\lambda-c\lambda^{-n}\leq(1-\epsilon)\lambda\leq-(1-\epsilon)\mu.$$

Thus,

$$\inf\{|\lambda|;\ \lambda\in\mathrm{Sp}\,Q_{\epsilon,c},\lambda\neq 0\}\geq(1-|\epsilon|)\mu.$$

Let us now choose a so that $0<a<(1-|\epsilon|)\mu$; this is always possible if ϵ and a are small enough. Then $Q_{\epsilon,c}$ has no non-zero eigenvalues in the interval $[-a,a]$. Therefore, using (3.6) we see that

$$F(Q_{\epsilon,c}+a)=F(Q_{\epsilon,c})+\Pi_0(Q_{\epsilon,c}),$$

and hence $P_{\epsilon,c,a}$ is equal to

$$\begin{aligned}\left(F(Q_{\epsilon,c})+\Pi_0(Q_{\epsilon,c})\right)^{m+1}(Q_{\epsilon,c}+a)^m&=\left(F(Q_{\epsilon,c})^{m+1}+\Pi_0(Q_{\epsilon,c})\right)(Q_{\epsilon,c}+a)^m\\&=F(Q_{\epsilon,c})^{m+1}(Q_{\epsilon,c}+a)^m+a^m\Pi_0(Q_{\epsilon,c}).\end{aligned}$$

As (4.12) and (5.2) show that $F(Q_{\epsilon,c})=F(Q)=F(P)$ and $\Pi_0(Q_{\epsilon,c})=\Pi_0(Q)=\Pi_0(P)$, we deduce that

$$P_{\epsilon,c,a}=F(P)^{m+1}(Q_{\epsilon,c}+a)^m+a^m\Pi_0(P).\tag{5.4}$$

As shown immediately above Corollary 4.4, when (ϵ,c,a) converges to $(0,0,0)$ the operator $Q_{\epsilon,c}+a$ converges to Q in $\Psi^1(M,E)$. Combining this with (5.2) and (5.4) shows that $P_{\epsilon,c,a}$ converges to $F(P)^{m+1}Q^m=P$. This proves the first part of the proposition and completes the proof. □

As immediate consequences of Proposition 5.2 we obtain

Corollary 5.3. *We always can perturbate P by an arbitrary small selfadjoint element of $\Psi^m(M,E)$ in such way that the eta function $\eta(P;s)$ and the zeta functions $\zeta_{\uparrow\downarrow}(P;s)$ and $\zeta(|P|;s)$ become singular at all points of the admissible set (5.1).*

Corollary 5.4. *Generically, the eta and zeta functions of selfadjoint elliptic ΨDOs of positive orders are singular at all points of the admissible set (5.1).*

Appendix A. Topologies on Spaces of ΨDOs

In this section, for reader's convenience we briefly recall the definition of the standard Fréchet space topology of the space $\Psi^m(M,E)$, $m\in\mathbb{C}$.

Let U be an open subset of $\mathbb{R}^n$. The space of classical symbols $S^m(U\times\mathbb{R}^n, M_r(\mathbb{C}))$ consists of functions $p(x,\xi)\in C^\infty(U\times\mathbb{R}^n, M_r(\mathbb{C}))$ with an asymptotic expansion,

$$p(x,\xi)\sim\sum_{j\geq 0}p_{m-j}(x,\xi), \tag{A.1}$$

where $p_{m-j}(x,\xi)\in C^\infty(\mathbb{R}^n\setminus 0, M_r(\mathbb{C}))$ is homogeneous of degree $m-j$ with respect to the variable ξ, i.e.,

$$p_{m-j}(x,\lambda\xi)=\lambda^{m-j}p_{m-j}(x,\xi)\qquad\forall\lambda>0,$$

and the asymptotics is taken in the sense that, for all compacts $K\subset U$, integers $N\geq 1$ and multi-orders α and β, there exists a constant $C_{KN\alpha\beta}>0$ such that, for all $x\in K$ and $\xi\in\mathbb{R}^n$ with $|\xi|\geq 1$, we have

$$\left|\partial_x^\alpha\partial_\xi^\beta\left(p(x,\xi)-\sum_{j<N}p_{m-j}(x,\xi)\right)\right|\leq C_{KN\alpha\beta}|\xi|^{m-N-|\beta|}.$$

We endow $S^m(U\times\mathbb{R}^n, M_r(\mathbb{C}))$ with the locally convex topological vector space topology defined by the semi-norms,

$$\mathfrak{p}_{KN}(p):=\sup_{|\alpha|+|\beta|\leq N}\ \sup_{(x,\xi)\in K\times\mathbb{R}^n}(1+|\xi|)^{-m+|\beta|}|\partial_x^\alpha\partial_\xi^\beta p(x,\xi)|$$

$$\mathfrak{q}_{KN}(p):=\sup_{|\alpha|+|\beta|\leq N}\ \sup_{\substack{x\in K\\|\xi|\geq 1}}(1+|\xi|)^{-m+|\beta|+N}\left|\partial_x^\alpha\partial_\xi^\beta\left(p(x,\xi)-\sum_{j<N}p_{m-j}(x,\xi)\right)\right|,$$

where K ranges over a (countable) compact exhaustion of U and N ranges over all positive integers. One can check that with respect to this topology $S^m(U\times\mathbb{R}^n, M_r(\mathbb{C}))$ is a Fréchet space.

Let $\tau: E_{|V}\xrightarrow{\sim}V\times\mathbb{C}^r$ be a local trivialization of E over an open $V\subset M$ which is the domain of a local chart $\kappa: V\xrightarrow{\sim}U\subset\mathbb{R}^n$. We then have pushforward and pullback maps,

$$\tau_*: C^\infty(V,E_{|V})\to C^\infty(V,\mathbb{C}^r)\qquad\text{and}\qquad\tau^*: C^\infty(V,\mathbb{C}^r)\to C^\infty(V,E_{|V}),$$

such that

$$\begin{aligned}\tau(u(x))&=(x,\tau_*u(x))\qquad\forall u\in C^\infty(V,E_{|V}),\\ \tau^*u(x)&=\tau^{-1}(x,u(x))\qquad\forall u\in C^\infty(V,\mathbb{C}^r).\end{aligned}$$

Let $P\in\Psi^m(M,E)$. In the local coordinates and trivialization defined by κ and τ, the operator P corresponds to the operator $P_{\kappa,\tau}\in\Psi^m(U,\mathbb{C}^r)$ such that

$$Pu(x)=\tau^*\left[P_{\kappa,\tau}\left((\tau_*u)\circ\kappa^{-1}\right)\right](\kappa(x))\qquad\forall u\in C_c^\infty(V,E)\ \forall x\in V.$$

Let φ and ψ be functions in $C_c^\infty(U)$. Then the Schwartz kernel of the operator $\varphi P_{\kappa,\tau}\psi$ has compact support, and hence $\varphi P_{\kappa,\tau}\psi$ is properly supported. Therefore, there exists a unique symbol $p_{\kappa,\tau}^{\varphi,\psi}(P)(x,\xi)\in S^m(U\times\mathbb{R}^n, M_r(\mathbb{C}))$ so that

$$\varphi(x)P_{\kappa,\tau}(\psi u)(x)=(2\pi)^{-n}\int e^{ix\cdot\xi}p_{\kappa,\tau}^{\varphi,\psi}(P)(x,\xi).\hat{u}(\xi)d\xi\qquad\forall u\in C_c^\infty(U,\mathbb{C}^r).$$

Namely, setting $e_\xi(x):=e^{ix\cdot\xi}$, we have

$$p_{\kappa,\tau}^{\varphi,\psi}(P)(x,\xi)(P)(x,\xi)=e^{-ix\cdot\xi}\varphi(x)P_{\kappa,\tau}(\psi e_\xi)(x)\qquad\forall(x,\xi)\in U\times\mathbb{R}^n.$$

We thus get a linear map,

$$\Psi^m(M,E) \ni P \longrightarrow p^{\varphi,\psi}_{\kappa,\tau}(P)(x,\xi) \in S^m(U \times \mathbb{R}^n, M_r(\mathbb{C})). \tag{A.2}$$

The topology of $\Psi^m(M,E)$ is the weakest locally convex topological vector space topology that makes continuous *all* the linear maps (A.2) as (κ,τ) ranges over all pairs where κ is a local chart for M and τ is a local trivialization of E over the domain of κ and the pair (φ,ψ) range over all pairs of functions in $C^\infty_c(U)$. With respect to this topology $\Psi^m(M,E)$ is a Fréchet space. In addition, as M is compact, one can check that the standard operations with ΨDOs, such that as products, taking adjoints or actions of diffeomorphisms, are all continuous with respect to this topology.

Appendix B. Corrections to Theorem 6.3 of [Po1]

As we shall explain Proposition 3.6 contradicts Theorem 6.3 of [Po1]. In this appendix, we state and prove the correct version of that statement.

Let $P : C^\infty(M,E) \to C^\infty(M,E)$ be a selfadjoint elliptic ΨDO of order m, $m \in \mathbb{N}$, which is odd-class, in the sense that, in any given local coordinates and trivialization of E, the homogeneous components of its symbol satisfy (3.10). The class of odd-class ΨDOs is a subalgebra of $\Psi^{\mathbb{Z}}(M,E)$. In particular, it is invariant under perturbations by constants. Moreover, any parametrix of an odd-class ΨDO is again an odd-class ΨDO. Therefore, for all $k \in \mathbb{Z}$, the operator P^{-k} is an odd-class ΨDO.

Theorem 6.3 of [Po1] asserts that if m and $n = \dim M$ have opposite parities, then the eta function $\eta(P;s)$ is regular at all integer points. If P has order 1 and n is even, then this implies that the eta function is actually entire. This assertion is clearly contradicted by Proposition 3.6.

As we shall now explain, the correct version of Theorem 6.3 of [Po1] is

Theorem B.1. (1) *If n is odd and m is even, then $\eta(P;s)$ is regular at all integer points.*

(2) *If n is even and m is odd, then $\eta(P;s)$ is regular at all even integer points.*

(3) *If n is even and $m = 1$, then the singular set of the function $\eta(P;s)$ only contains odd integers $\leq n$.*

In other words, Theorem 6.3 of [Po1] is true without modifications in odd dimension, but in even dimension we only have regularity at *even* integer points.

As it turns out, the caveat in the proof of Theorem 6.3 of [Po1] comes from using Proposition 6.2 of [Po1], which relates the eta function $\eta(P;s)$ to the zeta functions $\zeta_{\uparrow\downarrow}(P;s)$. Namely, by Eq. (6.2) of [Po1], for all $s \in \mathbb{C}$, we have

$$P^s_\uparrow - P^s_\downarrow = (1-e^{i\pi s})P^s_\uparrow - (1-e^{i\pi s})F|P|^s.$$

Therefore, we have the equality of meromorphic functions on $\mathbb{C}$,

$$\zeta_\uparrow(P;s) - \zeta_\downarrow(P;s) = (1-e^{-i\pi s})\zeta_\uparrow(P;s) - (1-e^{-i\pi s})\eta(P;s). \tag{B.1}$$

The above equality is the content of the first part of Proposition 6.4 of [Po1]. However, the other parts of this statement about the residues of the eta functions at integer points do not hold in full generality, but they do hold if we restrict ourselves to *even* integers. More precisely, we have

Proposition B.2. (1) *If k is an even integer, then*

$$m.\lim_{s\to k}(\zeta_\uparrow(P;s) - \zeta_\downarrow(P;s)) = i\pi \operatorname{Res} P^{-k} - i\pi m.\operatorname{res}_{s=k}\eta(P;s). \tag{B.2}$$

(2) *Let k be an even integer such that* $\operatorname{Res} P^{-k} = 0$*, so that both* $\zeta_\uparrow(P;s)$ *and* $\zeta_\downarrow(P;s)$ *are regular at* $s = k$*. Then*

$$\zeta_\uparrow(P;k) = \zeta_\downarrow(P;k) \Longleftrightarrow \eta(P;s) \text{ is regular at } s = k.$$

Proof. As the 2nd part is an immediate consequence of the first part, we only have to prove the latter. Suppose that k is an even integer. Then $1 - e^{-i\pi s} \sim i\pi(s-k)$ near $s = k$, and hence

$$\lim_{s\to k}(1 - e^{-i\pi s})\zeta_\uparrow(P;s) = i\pi \operatorname{Res}_{s=k} \zeta_\uparrow(P;s) = i\pi m^{-1} \operatorname{Res} P^{-k},$$

$$\lim_{s\to k}(1 - e^{-i\pi s})\eta(P;s) = i\pi \operatorname{Res}_{s=k} \eta(P;s).$$

Combining this with (B.1) then gives (B.2). This proves the first part and completes the proof. □

We are now in a position to prove Theorem B.1.

Proof of Theorem B.1. Suppose that n is odd and m is even, and let $k \in \mathbb{Z}$. As mentioned above P^{-k} is an odd-class ΨDO. Moreover, as alluded to in the proof of Lemma 3.5, in odd dimensions the noncommutative residue vanishes on all odd-class ΨDOs. In particular, we see that

$$\operatorname{Res} P^{-k} = 0. \tag{B.3}$$

Suppose that k is even. Then, by Theorem 5.1 of [Po1],

$$\lim_{s\to k}(\zeta_\uparrow(P;s) - \zeta_\downarrow(P;s)) = 0.$$

Combining this with (B.2) and (B.3) shows that $\eta(P;s)$ is regular at $s = k$ when k is even.

Assume that k is odd. Thanks to (2.8) we have

$$\operatorname{Res}_{s=k} \eta(P;s) = m^{-1} \operatorname{Res}\left[F|P|^{-k}\right] = m^{-1} \operatorname{Res}\left[F^{k+1}P^{-k}\right].$$

Moreover, as $k+1$ is even, Eq. (3.4) implies that $F^{k+1}P^{-k} = P^{-k} \bmod \Psi^{-\infty}(M,E)$. Thus,

$$\operatorname{Res}_{s=k} \eta(P;s) = m^{-1} \operatorname{Res} P^{-k} = 0.$$

This shows that $\eta(P;s)$ is regular at $s = k$, even when k is odd.

Suppose now that n is even and m is odd, and let k be an *even* integer. Thanks to Theorem 5.2 of [Po1] we have

$$m.\lim_{s\to k}(\zeta_\uparrow(P;s) - \zeta_\downarrow(P;s)) = i\pi \operatorname{Res} P^{-k}.$$

Combining this with (B.2) proves that $\operatorname{Res}_{s=k} \eta(P;s) = 0$, that is, the function $\eta(P;s)$ is regular at $s = k$.

If we further assume that P has order 1, then the admissible set (2.4) at which the function $\eta(P;s)$ may be singular only contains integers $\leq n$. Since we know that $\eta(P;s)$ is regular at all even integer points, we see that the singularities of $\eta(P;s)$ can only occur at odd integers $\leq n$. The proof of Theorem B.1 is thus complete. □

References

[ABP] Atiyah, M., Bott, R., Patodi, V.: *On the heat equation and the index theorem.* Invent. Math. **19**, 279–330 (1973). *Errata.* Invent. Math. **28**, 277–280 (1975).

[APS1] Atiyah, M.F.; Patodi, V.K.; Singer, I.M.: *Spectral asymmetry and Riemannian geometry. I.* Math. Proc. Camb. Philos. Soc. **77** (1975) 43–69 .

[APS2] Atiyah, M.F.; Patodi, V.K.; Singer, I.M.: Spectral asymmetry and Riemannian geometry. III. Math. Proc. Camb. Philos. Soc. **79** (1976) 71–99 .

[BG] Branson, T; Gilkey, P.B.: *Residues of the eta function for an operator of Dirac type.* J. Funct. Anal. **108** (1992), no. 1, 47–87.

[BS] Brüning, J.; Seeley, R.T.: *An index theorem for first order regular singular operators.* Amer. J. Math. **110** (1988) 659–714.

[Gi1] Gilkey, P.B.: *The residue of the global η function at the origin.* Adv. in Math. **40** (1981), no. 3, 290–307.

[Gi2] Gilkey, P.B.: *Invariance theory, the heat equation and the Atiyah-Singer index theorem.* 2nd edition, CRC Press, Boca Raton, 1994.

[Gu1] Guillemin, V.W.: *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues.* Adv. in Math. **55** (1985), no. 2, 131–160.

[Gu2] Guillemin, V.W.: *Gauged Lagrangian distributions.* Adv. Math. **102** (1993), no. 2, 184–201.

[Gu3] Guillemin, V.W.: *Residue traces for certain algebras of Fourier integral operators.* J. Funct. Anal. **115** (1993), no. 2, 391–417.

[Ka] Kastler, D.: *The Dirac operator and gravitation.* Comm. Math. Phys. **166** (1995), no. 3, 633–643.

[KV] Kontsevich, M.; Vishik, S.: *Geometry of determinants of elliptic operators.* Progr. Math., 131, pp. 173–197, Birkhäuser, 1995.

[LP] Lescure, J.M.; Paycha, S.: *Uniqueness of multiplicative determinants on elliptic pseudodifferential operators.* Proc. London Math. Soc. (3) **94** (2007) 772–812.

[Ok] Okikiolu, K.: *The Campbell-Hausdorff theorem for elliptic operators and a related trace formula.* Duke. Math. J. **79** (1995) 687–722.

[Po1] Ponge, R.: *Spectral asymmetry, zeta functions and the noncommutative residue.* Int. J. Math. **17** (2006) 1065–1090.

[Po2] Ponge, R.: *Noncommutative geometry and lower dimensional volumes in Riemannian geometry.* Lett. Math. Phys. **83** (2008) 19–32.

[Po3] Ponge, R.: *Traces on pseudodifferential operators and sums of commutators.* J. Anal. Math. **110** (2010) 1–30.

[Se] Seeley, R.T.: *Complex powers of an elliptic operator.* Proc. Sympos. Pure Math., Vol. X, pp. 288–307. AMS, Providence, R.I., 1967.

[Wo1] Wodzicki, M.: *Spectral asymmetry and noncommutative residue* (in Russian). Thesis, Steklov Institute, (former) Soviet Academy of Sciences, Moscow, 1984.

[Wo2] Wodzicki, M.: *Noncommutative residue. I. Fundamentals. K-theory, arithmetic and geometry* (Moscow, 1984–1986), 320–399, Lecture Notes in Math., 1289, Springer, 1987.

Department of Mathematics, SUNY Binghamton University, Binghamton, NY, USA
E-mail address: paul@math.binghamton.edu

Institutul de Matematică al Academiei Române, Bucharest, Romania
E-mail address: moroianu@alum.mit.edu

IPMU & Graduate School of Mathematical Sciences, University of Tokyo, Tokyo, Japan
E-mail address: ponge@ms.u-tokyo.ac.jp