ON THE STRUCTURE OF QUANTUM PERMUTATION GROUPS

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ABSTRACT. The quantum permutation group of the set $X_n = \{1, \ldots, n\}$ corresponds to the Hopf algebra $A_{aut}(X_n)$. This is an algebra constructed with generators and relations, known to be isomorphic to $\mathbb{C}(S_n)$ for $n \leq 3$, and to be infinite dimensional for $n \geq 4$. In this paper we find an explicit representation of the algebra $A_{aut}(X_n)$, related to Clifford algebras. For $n = 4$ the representation is faithful in the discrete quantum group sense.

INTRODUCTION

A general theory of unital Hopf $\mathbb{C}^*$-algebras is developed by Woronowicz in [11], [12], [13]. The main results are the existence of the Haar functional, an analogue of Peter-Weyl theory and of Tannaka-Krein duality, and explicit formulae for the square of the antipode. As for examples, these include algebras of continuous functions on compact groups, q-deformations of them with $q > 0$, and $\mathbb{C}^*$-algebras of discrete groups.

Of particular interest is the algebra $A_{aut}(X_n)$ constructed by Wang in [9]. This is the universal Hopf $\mathbb{C}^*$-algebra coacting on the set $X_n = \{1, \ldots, n\}$. In other words, the compact quantum group associated to it is a kind of analogue of the symmetric group S_n.

The algebra $A_{aut}(X_n)$ is constructed with generators and relations. There are n^2 generators, labeled u_{ij} with $i, j = 1, \ldots, n$. The relations are those making u a magic biunitary matrix. This means that all coefficients u_{ij} are projections, and on each row and each column of u these projections are mutually orthogonal, and sum up to 1. The comultiplication is given by $\Delta(u_{ij}) = \sum u_{ik} \otimes u_{kj}$ and the fundamental coaction is given by $\alpha(\delta_i) = \sum u_{ji} \otimes \delta_j$.

For $n = 1, 2, 3$ the canonical quotient map $A_{aut}(X_n) \to \mathbb{C}(S_n)$ is an isomorphism. For $n \geq 4$ the algebra $A_{aut}(X_n)$ is infinite dimensional, and just a few things are known about it. Its irreducible corepresentations are classified in [3], with the conclusion that their fusion rules coincide with those for irreducible representations of $SO(3)$, independently of $n \geq 4$. In [10] Wang proves that the compact quantum group associated to $A_{aut}(X_n)$ with $n \geq 4$ is simple. In [3] it is shown that the discrete quantum group associated to $A_{aut}(X_n)$ with $n \geq 5$ is not amenable. Various quotients of $A_{aut}(X_n)$, corresponding to quantum symmetry groups of polyhedra, colored graphs etc., are studied in [4] by using planar algebra techniques.

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These results certainly bring some light on the structure of $A_{aut}(X_n)$. However, for $n \geq 4$ this remains an abstract algebra, constructed with generators and relations.

In this paper we find an explicit representation of $A_{aut}(X_n)$. The construction works when n is a power of 2, and uses a magic biunitary matrix related to Clifford algebras. For $n = 4$ the representation is inner faithful, in the sense that the corresponding unitary representation of the discrete quantum group associated to $A_{aut}(X_4)$ is faithful.

As a conclusion, there might be a geometric interpretation of Hopf algebras of type $A_{aut}(X_n)$. We should mention here that for the algebra $A_{aut}(X)$ with X finite graph such an interpretation would be of real help, for instance in computing fusion rules.

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1. Magic biunitary matrices

Let A be a unital $\mathbb{C}^*$-algebra. That is, we have a unital algebra A over the field of complex numbers $\mathbb{C}$, with an antilinear antimultiplicative map $a \to a^*$ satisfying $a^{**} = a$, and with a Banach space norm satisfying $\|a^*a\| = \|a\|^2$.

A projection is an element $p \in A$ satisfying $p^2 = p^* = p$. Two projections p, q are said to be orthogonal when $pq = 0$. A partition of unity in A is a finite set of projections, which are mutually orthogonal, and sum up to 1.

Definition 1.1. A matrix $v \in M_n(A)$ is called magic biunitary if all its rows and columns are partitions of the unity of A.

A magic biunitary is indeed a biunitary, in the sense that both v and its transpose v^t are unitaries. The other word – magic – comes from a vague similarity with magic squares.

The basic example comes from the symmetric group S_n. Consider the sets of permutations $\{\sigma \in S_n \mid \sigma(j) = i\}$. When i is fixed and j varies, or vice versa, these sets form partitions of S_n. Thus their characteristic functions $v_{ij} \in \mathbb{C}(S_n)$ form a magic biunitary.

Of particular interest is the "universal" magic biunitary matrix. This has coefficients in the universal algebra $A_{aut}(X_n)$ constructed by Wang in [9].

Definition 1.2. $A_{aut}(X_n)$ is the universal $\mathbb{C}^*$-algebra generated by n^2 elements u_{ij}, subject to the magic biunitarity condition.

In other words, we have the following universal property. For any magic biunitary matrix $v \in M_n(A)$ there is a morphism of $\mathbb{C}^*$-algebras $A_{aut}(X_n) \to A$ mapping $u_{ij} \to v_{ij}$.

A more elaborate version of this property, to be discussed now, states that $A_{aut}(X_n)$ is a Hopf $\mathbb{C}^*$-algebra, whose underlying quantum group is a kind of analogue of S_n.

The following definition is due to Woronowicz [13].

Definition 1.3. A unital Hopf $\mathbb{C}^*$-algebra is a unital $\mathbb{C}^*$-algebra A, together with a morphism of $\mathbb{C}^*$-algebras $\Delta : A \to A \otimes A$, subject to the following conditions.

(i) Coassociativity condition: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.

(ii) Cocancellation condition: $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$.

The basic example is the algebra $\mathbb{C}(G)$ of continuous functions on a compact group G, with $\Delta(\varphi) : (g, h) \to \varphi(gh)$. Here coassociativity of Δ follows from associativity of the multiplication of G, and cocancellation in $\mathbb{C}(G)$ follows from cancellation in G.

Another example is the group algebra $\mathbb{C}^*(\Gamma)$ of a discrete group Γ. This is obtained from the usual group algebra $\mathbb{C}[\Gamma]$ by a standard completion procedure. The comultiplication is defined on generators $g \in \Gamma$ by the formula $\Delta(g) = g \otimes g$.

In general, associated to a Hopf $\mathbb{C}^*$-algebra A are a compact quantum group G and a discrete quantum group Γ, according to the heuristic formula $A = \mathbb{C}(G) = \mathbb{C}^*(\Gamma)$.

Definition 1.4. A coaction of A on a finite set X is a morphism of $\mathbb{C}^*$-algebras $\alpha : \mathbb{C}(X) \to \mathbb{C}(X) \otimes A$, subject to the following conditions.
(i) Coassociativity condition: $(\alpha \otimes id)\alpha = (id \otimes \Delta)\alpha$.
(ii) Natural condition: $(\Sigma \otimes id)v = \Sigma(.)1$, where $\Sigma(\varphi)$ is the sum of values of φ.

The basic example is with a group G of permutations of X. Consider the action map $a : X \times G \to X$, given by $a(i, \sigma) = \sigma(i)$. The formula $\alpha\varphi = \varphi a$ defines a morphism of $\mathbb{C}^*$-algebras $\alpha : \mathbb{C}(X) \to \mathbb{C}(X \times G)$. This can be regarded as a coaction of $\mathbb{C}(G)$ on X.

In general, coactions of A can be thought of as coming from actions of the underlying compact quantum group G. With this interpretation, the natural condition says that the action of G must preserve the counting measure on X. This assumption cannot be dropped.

The following fundamental result is due to Wang [9].

Theorem 1.1. *(i) $A_{aut}(X_n)$ is a Hopf $\mathbb{C}^*$-algebra, with comultiplication $\Delta(u_{ij}) = \sum u_{ik} \otimes u_{kj}$.*
(ii) The linear map $\alpha(\delta_j) = \sum \delta_i \otimes u_{ji}$ is a coaction of $A_{aut}(X_n)$ on $X_n = \{1, \ldots, n\}$.
(iii) $A_{aut}(X_n)$ is the universal Hopf $\mathbb{C}^*$-algebra coacting on X_n.

The idea for proving (i) is that we can define Δ by using the universal property of $A_{aut}(X_n)$. Coassociativity is clear, and cocancellation follows from a result of Woronowicz in [13], stating that this is automatic whenever there is a counit and an antipode. But these can be defined by $\varepsilon(u_{ij}) = \delta_{ij}$ and $S(u_{ij}) = u_{ji}$, once again by using universality of $A_{aut}(X_n)$.

We know that the compact quantum group G_n associated to $A_{aut}(X_n)$ is a kind of quantum analogue of the symmetric group S_n. In particular there should be an inclusion $S_n \subset G_n$. Here is the exact formulation of this observation, see Wang [9] for details.

Proposition 1.1. *There is a Hopf $\mathbb{C}^*$-algebra morphism $\pi_n : A_{aut}(X_n) \to \mathbb{C}(S_n)$, mapping the generators u_{ij} to the characteristic functions of the sets $\{\sigma \in S_n \mid \sigma(j) = i\}$.*

The question is now whether π_n is an isomorphism or not. For instance a 2×2 magic biunitary must be of the following special form, where p is a projection.

$$\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

The algebra generated by p is canonically isomorphic to $\mathbb{C}^2$ if $p \neq 0, 1$, and to $\mathbb{C}$ if not. Thus the universal algebra $A_{aut}(X_2)$ is isomorphic to $\mathbb{C}^2$, and π_2 is an isomorphism.

The map π_3 is an isomorphism as well, see [4] for a proof.

At $n = 4$ we have the following example of magic biunitary matrix.

$$\begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

We can choose the projections p, q such that the algebra $< p, q >$ they generate is infinite dimensional and not commutative. It follows that $A_{aut}(X_4)$ is infinite dimensional and not commutative as well, so π_4 cannot be an isomorphism.

Proposition 1.2. *For $n \geq 4$ the algebra $A_{aut}(X_n)$ is infinite dimensional and not commutative. In particular π_n is not an isomorphism.*

This follows by gluing an identity matrix of size $n - 4$ to the above 4×4 matrix.

There is a quantum group interpretation here. Consider the compact and discrete quantum groups defined by the formula $A_{aut}(X_4) = \mathbb{C}(G_4) = \mathbb{C}^*(\Gamma_4)$. When p, q are free the surjective morphism of $\mathbb{C}^*$-algebras $A_{aut}(X_4) \to < p, q >$ can be thought of as coming from a surjective morphism of discrete quantum groups $\Gamma_4 \to \mathbb{Z}_2 * \mathbb{Z}_2$. This makes it clear that Γ_4 is infinite. Now G_4 being the Pontrjagin dual of Γ_4, it must be infinite as well.

See Bichon [5], Wang [9], [10] and [4] for further speculations on this subject.

2. Inner faithful representations

We would like to find an explicit representation of $A_{aut}(X_n)$. As with any Hopf $\mathbb{C}^*$-algebra, there is a problem here, because there are two notions of faithfulness.

Consider for instance a discrete subgroup Γ of the unitary group $U(n)$. The inclusion $\Gamma \subset U(n)$ can be regarded as a unitary group representation $\Gamma \to U(n)$, and we get a $\mathbb{C}^*$-algebra representation $\mathbb{C}^*(\Gamma) \to M_n(\mathbb{C})$. This latter representation is far from being faithful: for instance its kernel is infinite dimensional, hence nonempty, when Γ is an infinite group. However, the representation $\mathbb{C}^*(\Gamma) \to M_n(\mathbb{C})$ must be "inner faithful" in some Hopf $\mathbb{C}^*$-algebra sense, because the representation $\Gamma \to U(n)$ it comes from is faithful.

So, we are led to the following question. Let H be a unital Hopf $\mathbb{C}^*$-algebra, and let $\pi : H \to A$ be a morphism of $\mathbb{C}^*$-algebras. If Γ is the discrete quantum group associated to H we know that π corresponds to a unitary representation $\pi_i : \Gamma \to U(A)$. The question is: when is π inner faithful, meaning that π_i is faithful?

A simple answer is obtained by using the formalism of Kustermans and Vaes [6]. Associated to H is a von Neumann algebra H_{vN}, obtained by a certain completion procedure. Now coefficients of π belong to the dual algebra $\widehat{H}_{vN}$, and we can say that π is inner faithful if these coefficients generate $\widehat{H}_{vN}$. This notion is used by Vaes in [7], and a version of it is used by Wang in [9].

In this paper we use an equivalent definition, from [2].

Definition 2.1. Let H be a unital Hopf $\mathbb{C}^*$-algebra. A $\mathbb{C}^*$-algebra representation $\pi : H \to A$ is called inner faithful if the $*$-algebra generated by its coefficients is dense in H^*_{alg}.

Here H_{alg} is the dense $*$-subalgebra of H consisting of "representative functions" on the underlying compact quantum group, constructed by Woronowicz in [13]. This is a Hopf $*$-algebra in the usual sense. Its dual complex vector space H^*_{alg} is a $*$-algebra, with multiplication Δ^* and involution $*^*$. Finally, coefficients of π are the linear forms $\varphi\pi$ with $\varphi \in A^*$, and the density assumption is with respect to the weak topology on H^*_{alg}. See e.g. the book of Abe [1] for Hopf algebras and [2] for details regarding this definition.

The main example is with a discrete group Γ. As expected, a representation $\mathbb{C}^*(\Gamma) \to A$ is inner faithful if and only if the corresponding unitary group representation $\Gamma \to U(A)$ is faithful. Some other examples are discussed in [2].

For how to use inner faithfulness see Vaes [7].

Definition 2.2. The character of a magic biunitary matrix $v \in M_n(A)$ is the sum of its diagonal entries $\chi(v) = v_{11} + v_{22} + \ldots + v_{nn}$.

The terminology comes from the case where $v = u$ is the universal magic biunitary matrix, with coefficients in $A = A_{aut}(X_n)$. Indeed, the matrix u is a corepresentation of $A_{aut}(X_n)$ in the sense of Woronowicz [11], and the element $\chi(u)$ is its character.

Lemma 2.1. *Let $v \in M_n(A)$ be a magic biunitary matrix, with $n \geq 4$. Assume that there is a unital linear form $\varphi : A \to \mathbb{C}$ such that*

$$\varphi\left(\chi(v)^k\right) = \frac{1}{k+1}\binom{2k}{k}$$

for any k. Then the representation $\pi : A_{aut}(X_n) \to A$ defined by $u_{ij} \to v_{ij}$ is inner faithful.

Proof. The numbers in the statement are the Catalan numbers, appearing as multiplicities in representation theory of $SO(3)$. The result will follow from the following fact from [3]. The finite dimensional irreducible corepresentations of $A_{aut}(X_n)$ can be arranged in a sequence $\{r_k\}$, such that their fusion rules are the same as those for representations of $SO(3)$.

$$r_k \otimes r_s = r_{|k-s|} + r_{|k-s|+1} + \ldots + r_{k+s}$$

Let $h : A_{aut}(X_n) \to \mathbb{C}$ be the Haar functional, constructed by Woronowicz in [11]. Consider also the character of the fundamental corepresentation of $A_{aut}(X_n)$.

$$\chi(u) = u_{11} + u_{22} + \ldots + u_{nn}$$

The Poincaré series of $A_{aut}(X_n)$ is defined by the following formula.

$$f(z) = \sum_{k=0}^{\infty} h\left(\chi(u)^k\right) z^k$$

By the above result, this is equal to the Poincaré series for $SO(3)$.

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k} z^k$$

The assumption of the lemma says that the equality $\varphi\pi = h$ holds on all powers of $\chi(u)$. By linearity, this equality must hold on the algebra $< \chi(u) >$ generated by $\chi(u)$. Now by positivity of h it follows that the restriction of π to this algebra $< \chi(u) >$ is injective.

On the other hand, once again from fusion rules, we see that $\chi(u)$ generates the algebra of characters $A_{aut}(X_n)_{central}$ constructed by Woronowicz in [11].

Summing up, we know that π is faithful on $A_{aut}(X_n)_{central}$.

Consider now the "minimal model" construction in [2]. This is the factorisation of π into a Hopf $\mathbb{C}^*$-algebra morphism $A_{aut}(X_n) \to H$, and an inner faithful representation $H \to A$.

$$A_{aut}(X_n) \to H \to A$$

Since π is faithful on $A_{aut}(X_n)_{central}$, so is the map on the left. By Woronowicz's analogue of the Peter-Weyl theory in [11] it follows that the map on the left is an isomorphism. Thus π coincides with the map on the right, which is by definition inner faithful. □

It is possible to reformulate this result, by using notions from Voiculescu's free probability theory [8]. A non-commutative $\mathbb{C}^*$-probability space is a pair (A, φ) consisting of a unital $\mathbb{C}^*$-algebra A together with a positive unital linear form $\varphi : A \to \mathbb{C}$.

Associated to a self-adjoint element $x \in A$ is its spectral measure μ_x. This is a probability measure on the spectrum of x, defined by the formula

$$\varphi(f(x)) = \int_{\mathbb{R}} f(t)\, d\mu_x(t).$$

This equality must hold for any continuous function f on the spectrum of x. By density we can restrict attention to polynomials $f \in \mathbb{C}[X]$, then by linearity it is enough to have this equality for monomials $f(t) = t^k$. We say that μ_x is uniquely determined by its moments,

$$\varphi(x^k) = \int_{\mathbb{R}} t^k\, d\mu_x(t).$$

The following notion plays a central role in free probability. See [8], page 26.

Definition 2.3. An element x in a non-commutative $\mathbb{C}^*$-probability space is called semicircular if its spectral measure is $d\mu_x(t) = (2\pi)^{-1}\sqrt{4-t^2}\, dt$ on $[-2, 2]$, and 0 elsewhere.

In terms of moments, we must have the following equalities, for any k:

$$\varphi(x^k) = \frac{1}{2\pi}\int_{-2}^{2} t^k \sqrt{4-t^2}\, dt.$$

The integral is 0 when k is odd, and equal to a Catalan number when k is even,

$$\varphi(x^{2k}) = \frac{1}{k+1}\binom{2k}{k}.$$

We get in this way a reformulation of the above lemma.

Theorem 2.1. *A magic biunitary matrix whose character has same spectral measure as the square of a semicircular element produces an inner faithful representation of Wang's algebra.*

The assumption $n \geq 4$ was removed, because it is superfluous. Indeed, for $n = 1, 2, 3$ finite dimensionality of $A_{aut}(X_n)$ implies that the spectrum of any $\chi(v)$ is discrete.

3. Geometric constructions

Consider the Pauli matrices.

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

These satisfy the relations for quaternions $i^2 = j^2 = k^2 = -1$, $ij = ji = -k$ etc. To any $x \in SU(2)$ we associate the following matrix.

$$\begin{pmatrix} 1 \\ i \\ j \\ k \end{pmatrix} x \begin{pmatrix} 1 & i & j & k \end{pmatrix} = \begin{pmatrix} x & xi & xj & xk \\ ix & ixi & ixj & ixk \\ jx & jxi & jxj & jxk \\ kx & kxi & kxj & kxk \end{pmatrix}$$

Each row and each column of this matrix is an orthogonal basis of $M_2(\mathbb{C}) \simeq \mathbb{C}^4$ with respect to the inner product

$$\langle x, y \rangle = \frac{1}{2} Tr(xy^*),$$

since i, j, k are sqew-adjoints. Thus the matrix of corresponding orthogonal projections is a magic biunitary.

Theorem 3.1. *There is an inner faithful representation*

$$\pi : A_{aut}(X_4) \to \mathbb{C}(SU(2), M_4(\mathbb{C}))$$

mapping the universal 4×4 *magic biunitary matrix to the* 4×4 *matrix*

$$v(x) = \begin{pmatrix} P_x & P_{xi} & P_{xj} & P_{xk} \\ P_{ix} & P_{ixi} & P_{ixj} & P_{ixk} \\ P_{jx} & P_{jxi} & P_{jxj} & P_{jxk} \\ P_{kx} & P_{kxi} & P_{kxj} & P_{kxk} \end{pmatrix}$$

where for $y \in SU(2)$ *we denote by* P_y *the orthogonal projection onto the space* $\mathbb{C}y \subset M_2(\mathbb{C})$*, and we regard it as a continuous function of* y*, with values in* $M_2(M_2(\mathbb{C})) \simeq M_4(\mathbb{C})$*.*

Proof. We have to compute the character of $v = v(x)$.

$$\chi(v) = P_x + P_{ixi} + P_{jxj} + P_{kxk}$$

We make the convention that Greek letters designate quaternions in $\{1, i, j, k\}$. We decompose x as a sum with real coefficients $x = \sum x_\alpha \alpha$. We have the following formula for $\chi(v)$.

$$\chi(v) = \sum_\alpha P_{\alpha x \alpha}$$

With the notations $\alpha\beta = (-1)^{N(\alpha,\beta)}\beta\alpha$ and $\alpha^2 = (-1)^{N(\alpha)}$ we can compute $\alpha x \alpha$.

$$\alpha x \alpha = \sum_\beta (-1)^{N(\alpha,\beta)+N(\alpha)} x_\beta \beta$$

Now using the above-mentioned canonical scalar product on $M_2(\mathbb{C})$, this gives the following formula for $P_{\alpha x\alpha}$, after cancelling the $(-1)^{2N(\alpha)}=1$ term.

$$< P_{\alpha x\alpha}\beta,\gamma >= (-1)^{N(\alpha,\beta)+N(\alpha,\gamma)}x_\beta x_\gamma$$

Now summing over α gives the formula of the character $\chi(v)$.

$$< \chi(v)\beta,\gamma >= \sum_\alpha (-1)^{N(\alpha,\beta)+N(\alpha,\gamma)}x_\beta x_\gamma$$

The coefficient of $x_\beta x_\gamma$ can be computed by using the multiplication table of quaternions.

$$\sum_\alpha (-1)^{N(\alpha,\beta)+N(\alpha,\gamma)} = 4\delta_{\beta,\gamma}$$

Thus $\chi(v)$ is a diagonal matrix, having the numbers $4x_\beta^2$ on the diagonal.

$$\chi(v) = \mathrm{diag}(4x_\beta^2)$$

Consider the linear form $\varphi = \int \otimes\, tr$, where the integral is with respect to the Haar measure of $SU(2)$, and tr is the normalised trace of 4×4 matrices, meaning $1/4$ times the usual trace. The moments of $\chi(v)$ with respect to φ are computed as follows.

$$\int tr(\chi(v)^k)dx = 4^{k-1}\sum_\beta \int x_\beta^{2k}dx$$

By symmetry reasons the four integrals are all equal, say to the first one.

$$\int tr(\chi(v)^k)dx = 4^k \int x_1^{2k}dx$$

It follows that $\chi(v)$ has the same spectral measure as $4x_1^2$.

$$\mu_{\chi(v)} = \mu_{4x_1^2}$$

But the variable $2x_1$ is semicircular. This can be seen in many ways, for instance by direct computation, after identifying $SU(2)$ with the real sphere S^3, or by using the fact that $2x_1 = \mathrm{Tr}(x)$ is the character of the fundamental representation of $SU(2)$, whose moments are computed using Clebsch-Gordon rules. The result follows now by applying theorem 2.1. □

The construction of π has the following generalisation. Consider the Clifford algebra $Cl(\mathbb{R}^s)$. This is a finite dimensional algebra, having a basis formed by products $e_{i_1}\ldots e_{i_k}$ with $1\leq i_1<\ldots<i_k\leq s$, with multiplicative structure given by $e_i^2=-1$ and $e_ie_j=-e_je_i$ for $i\neq j$.

It is convenient to use the notation $e_I = e_{i_1}\ldots e_{i_k}$ with $I=(i_1,\ldots,i_k)$.

As an example, the Clifford algebra $Cl(\mathbb{R}^2)$ is spanned by the elements $e_\emptyset = 1$, e_1, e_2 and $e_{12}=e_1e_2$. The generators e_1, e_2 are subject to the relations $e_1^2=e_2^2=-1$ and $e_1e_2=-e_2e_1$. Now these relations are satisfied by the Pauli matrices i,j, and the corresponding representation of $Cl(\mathbb{R}^2)$ turns to be faithful. That is, we have the following identifications.

$$e_\emptyset = \begin{pmatrix}1&0\\0&1\end{pmatrix}\qquad e_1=\begin{pmatrix}i&0\\0&-i\end{pmatrix}\qquad e_2=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\qquad e_{12}=\begin{pmatrix}0&i\\i&0\end{pmatrix}$$

We can label as well indices of 4×4 matrices by elements of the set $\{\emptyset,1,2,12\}$. With these notations, the representation in theorem 3.1 is given by $\pi(u_{IJ}) = P_{e_Ixe_J}$.

The same formula works for an arbitrary number s.

Theorem 3.2. *There is a representation $\pi_n : A_{aut}(X_n) \to \mathbb{C}(G_n, M_n(\mathbb{C}))$ mapping the universal $n \times n$ magic biunitary matrix to the $n \times n$ matrix*

$$v = (P_{e_I x e_J})_{IJ}$$

where $n = 2^s$, the unitary group of the Clifford algebra $Cl(\mathbb{R}^s)$ is denoted G_n, and the algebra of endomorphisms of $Cl(\mathbb{R}^s)$ is identified with $M_n(\mathbb{C})$.

The first part of proof of theorem 3.1 extends to this general situation. We get that $\chi(v)$ is diagonal, with eigenvalues $\{nx_I^2\}$. This doesn't seem to be related to semicircular elements when $s \geq 3$. The representation π_n probably comes from an inner faithful representation of a quotient of $A_{aut}(X_n)$, corresponding to a "subgroup" of the quantum permutation group.

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