On Carvalho's K-theoretic formulation of the cobordism invariance of the index

Sergiu Moroianu

Abstract

We give a direct proof of the fact that the index of an elliptic operator on the boundary of a compact manifold vanishes when the principal symbol comes from the restriction of a K-theory class from the interior. The proof uses noncommutative residues inside the calculus of cusp pseudodifferential operators of Melrose.

Introduction

A classical result of Thom states that the topological signature of the boundary of a compact manifold with boundary vanishes. Viewing the signature as the index of an elliptic operator, Atiyah and Singer [2] generalized this vanishing to the so-called twisted signatures. This property, called the cobordism invariance of the index, was the essential step in their first proof of the index formula on closed manifolds. Conversely, cobordism invariance follows from the index theorem of [2].

On open manifolds a satisfactory index formula is not available, and probably not reasonable to expect in full generality. Such formulae in various particular cases are given e.g., in [1], [12] for manifolds with boundary, in [19], [9] for manifolds with fibered boundary, and in [11], [8] for manifolds with corners in the sense of Melrose. To advance in this direction, we believe it is important to understand conditions which ensure the vanishing of the index, in particular cobordism invariance, without using any index formula.

Direct proofs of the cobordism invariance of the index for first-order differential operators on closed manifolds were given e.g., in [4], [7], [10], [18], and also [17, Theorem 1]. We have proposed in [17] an extension of cobordism invariance to manifolds with corners. The result states that the sum of the indices on the hyperfaces is null, under suitable hypothesis.

All these results are partial, in that they only apply to differential operators of a special type. A well-known fact states that the index of "geometrically defined" operators is cobordism-invariant; but besides being vague, this is also not true (look at the Gauß-Bonnet operator). Only very recently, Carvalho [5, 6] found a remarkable K-theoretic statement of the cobordism invariance of the index using the topological approach of [3]. Here is a reformulation of the main result of [5] specialized to compact manifolds:

Theorem. *Let M be the boundary of the compact manifold X and D an elliptic pseudodifferential operator on M. The principal symbol of D defines a vector bundle over the sphere bundle inside $T^*M \oplus \mathbb{R}$. If the class in $K^0(S(T^*M \oplus \mathbb{R}))$ of this bundle is the restriction of a class from $K^0(S^*X)$ modulo $K^0(M)$, then* $\operatorname{index}(D) = 0$.

The missing details appear in Theorem 3. The aim of this paper is to reprove Theorem 3 by

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analytical methods, via the cusp calculus of pseudodifferential operators of Melrose on the manifold with boundary X.

In order to make the proof likely to generalize to open manifolds, we have made a point of avoiding to use results from K-theory, e.g., Bott periodicity and the index theorem. The proof will be based on Theorem 2, a statement about cusp pseudodifferential operators. One could try to use only *differential* operators and [17, Theorem 1], by representing any rational K-theory class on T^*M as the class of a first-order elliptic differential operator. Unfortunately, this is not possible if M is odd-dimensional since then the index of differential operators vanishes. Using a product with S^1 and arguing that the index is multiplicative leads to a vicious circle; indeed, this argument is enough for the proof of the index theorem [3], which we want to avoid.

1. Review of Melrose's cusp algebra

In this section we recall the facts about the cusp algebra needed in the sequel. For a full treatment of the cusp algebra we refer to [15] and [8].

Let X be a compact manifold with boundary M, and $x : X \to \mathbb{R}_+$ a boundary-defining function. A vector field V on X is called *cusp* if $dx(V) \in x^2\mathcal{C}^\infty(X)$. The space of cusp vector fields forms a Lie subalgebra ${}^c\mathcal{V}(X) \hookrightarrow \mathcal{V}(X)$ which is a finitely generated projective $\mathcal{C}^\infty(X)$-module; indeed, in a product decomposition $M \times [0,\epsilon) \hookrightarrow X$, a local basis of ${}^c\mathcal{V}(X)$ is given by $\{x^2\partial_x, \partial_{y_j}\}$ where y_j are local coordinates on M. Thus there exists a vector bundle ${}^cTX \to X$ such that ${}^c\mathcal{V}(X) = \mathcal{C}^\infty(X, {}^cTX)$.

The algebra $\mathcal{D}_c(X)$ of (scalar) cusp differential operators is defined as the universal enveloping algebra of ${}^c\mathcal{V}(X)$ over $\mathcal{C}^\infty(X)$. In a product decomposition as above, an operator in $\mathcal{D}_c(X)$ of order m takes the form

$$P = \sum_{j=0}^{m} P_{m-j}(x)(x^2\partial_x)^j \tag{1}$$

where $x \mapsto P_{m-j}(x)$ is a smooth family of differential operators of order $m-j$ on M.

1.1 Cusp pseudodifferential operators

The operators in $\mathcal{D}_c(X)$ can be described alternately (see [15]) in terms of their Schwartz kernels. Namely, there exists a manifold with corners X_c^2 obtained by blow-up from $X \times X$, and a submanifold Δ_c, such that $\mathcal{D}_c(X)$ corresponds to the space of distributions on X_c^2 which are classical conormal to Δ_c, supported on Δ_c and smooth at the boundary face of X_c^2 intersected by Δ_c. It is then a straightforward application of Melrose's program [13] to construct a calculus of pseudodifferential operators $\Psi_c^\lambda(X)$, $\lambda \in \mathbb{C}$, in which $\mathcal{D}_c(X)$ sits as the subalgebra of differential operators (the symbols used in the definition are classical of order λ). No difficulty appears in defining cups operators acting between sections of vector bundles over X. By composing with the multiplication operators by x^z, $z \in \mathbb{C}$, we get a pseudodifferential calculus with two complex indices

$$\Psi_c^{\lambda,z}(X, \mathcal{F}, \mathcal{G}) := x^{-z}\Psi_c^\lambda(X, \mathcal{F}, \mathcal{G})$$

such that $\Psi_c^{\lambda,z}(X, \mathcal{E}, \mathcal{F}) \subset \Psi_c^{\lambda',z'}(X, \mathcal{E}, \mathcal{F})$ if and only if $\lambda' - \lambda \in \mathbb{N}$ and $z' - z \in \mathbb{N}$ (since we work with classical symbols). Also,

$$\Psi_c^{\lambda,z}(X, \mathcal{G}, \mathcal{H}) \circ \Psi_c^{\lambda',z'}(X, \mathcal{F}, \mathcal{G}) \subset \Psi_c^{\lambda+\lambda',z+z'}(X, \mathcal{F}, \mathcal{H}).$$

By closure, cusp operators act on a scale of weighted Sobolev spaces $x^\alpha H_c^\beta$:

$$\Psi_c^{\lambda,z}(X, \mathcal{F}, \mathcal{G}) \times x^\alpha H_c^\beta(X, \mathcal{F}) \to x^{\alpha-\Re(z)} H_c^{\beta-\Re(\lambda)}(X, \mathcal{G}).$$

1.2 Symbol maps

A cusp differential operator of positive order cannot be elliptic at $x=0$ since its principal symbol will vanish on the normal covector to the boundary. Nevertheless, there exists a natural *cusp principal symbol* map surjecting onto the space of homogeneous functions on ${}^cT^*X \setminus \{0\}$ of homogeneity k:

$$\sigma : \Psi_c^k(X,\mathcal{E},\mathcal{F}) \to \mathcal{C}^\infty_{[k]}({}^cT^*X,\mathcal{E},\mathcal{F}).$$

In the sequel we refer to σ as the principal symbol map. A cusp operator is called elliptic if its principal symbol is invertible on ${}^cT^*X \setminus \{0\}$.

The second symbol map, the so-called indicial family, associates to any cusp operator $P \in \Psi_c(X,\mathcal{E},\mathcal{F})$ a family of pseudodifferential operators on M with one real parameter ξ. If P is given by (1) near $x=0$, then

$$I_M(P)(\xi) = \sum_{j=0}^m P_{m-j}(x)(i\xi)^j.$$

Elliptic cusp operators whose indicial family is invertible for each $\xi \in \mathbb{R}$ are called fully elliptic. By a general principle [12], for cusp operators being fully elliptic is equivalent to being Fredholm.

The principal symbol map and the indicial family are star-morphisms, i.e., they are multiplicative and commute with taking adjoints. The indicial family surjects onto the space $\Psi_{\mathrm{sus}}(M)$ of 1-suspended pseudodifferential operators defined in [14], that is, of families of operators on M with one real parameter ξ, with joint symbolic behavior in ξ and in the cotangent variables of T^*M.

Lemma 1. *Let $P \in \Psi^{-\infty}(M)$, and let $\phi \in \mathcal{S}(\mathbb{R})$ be a smooth compactly supported function. Then $\xi \mapsto \phi(\xi)P$ belongs to $\Psi^{-\infty}_{\mathrm{sus}}(M)$.*

Proof. The Schwartz kernel $\kappa(t,t',y,y') := \hat{\phi}(t-t')\kappa_P(y,y')$ is smooth and rapidly vanishing away from $t=t'$, so it defines a suspended operator of order $-\infty$ (see [14, Definition 1]). $\square$

1.3 Analytic families of cusp operators

Let $Q \in \Psi_c^{1,0}(X,\mathcal{E})$ be a positive fully elliptic cusp operator of order 1. Then the complex powers Q^λ form an analytic family of cusp operators of order λ.

Let $\mathbb{C}^2 \ni (\lambda,z) \mapsto P(\lambda,z) \in \Psi_c^{\lambda,z}(X,\mathcal{E})$ be an analytic family in two complex variables. Then $P(\lambda,z)$ is of trace class (on $L^2_c(M,\mathcal{E})$) for $\Re(\lambda) < -\dim(X), \Re(z) < -1$. Moreover, $(\lambda,z) \mapsto \mathrm{Tr}(P(\lambda,z))$ is analytic, extends to $\mathbb{C}^2$ meromorphically with at most simple poles in each variable at $\lambda \in \mathbb{N} - \dim(X)$, $z \in \mathbb{N} - 1$, and

$$\mathrm{Res}_{z=-1}\mathrm{Tr}(P(\lambda,z)) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}(I_M(x^{-1}P(\lambda,-1)))d\xi. \tag{2}$$

Indeed, this is the content of [17, Prop. 3].

2. Cobordism invariance for cusp operators

Theorem 2. *Let M be a closed manifold and*

$$D : \mathcal{C}^\infty(M,\mathcal{E}^+) \to \mathcal{C}^\infty(M,\mathcal{E}^-)$$

a classical pseudodifferential operator of order 1 on M. Assume that there exist hermitian vector bundles $V^+, V^- \to M$, $\mathcal{E} \to X$ with $\mathcal{E}|_M = \mathcal{E}^+ \oplus \mathcal{E}^- \oplus V^+ \oplus V^-$, and an elliptic symmetric cusp *pseudodifferential operator $A \in \Psi_c^{1,0}(X, \mathcal{E})$ such that*

$$I_M(A)(\xi) = \begin{bmatrix} \xi & \tilde{D}^*(\xi) & & \\ \tilde{D}(\xi) & -\xi & & \\ & & (1+\xi^2+\Delta^+)^{\frac{1}{2}} & \\ & & & -(1+\xi^2+\Delta^-)^{\frac{1}{2}} \end{bmatrix} \tag{3}$$

where Δ^+, Δ^- are the Laplacians of some connections on V^+, V^-, $\tilde{D} \in \Psi^1_{\text{sus}}(M, \mathcal{E}^+, \mathcal{E}^-)$ and $\tilde{D}(0) = D$. Then $\operatorname{index}(D) = 0$.

Proof. We first show that we can assume without loss of generality that D is either injective or surjective. Assuming this, we construct from A a positive cusp operator Q of order 1. The complex powers of Q are used in defining a complex number N as a non-commutative residue. The proof will be finished by computing N in two ways; first we get $N = 0$, then N is shown to be essentially $\operatorname{index}(D)$.

Reduction to the case where D is injective or surjective

Let $T \in \Psi^{-\infty}(M, \mathcal{E}^+, \mathcal{E}^-)$ be such that $D + T$ is either injective or surjective (or both). Choose $\tilde{T} \in \Psi^{-\infty}_{\text{sus}}(X, \mathcal{E}^+, \mathcal{E}^-)$ with $\tilde{T}(0) = T$. Choose $S \in \Psi_c^{-\infty,0}(X, \mathcal{E})$ such that

$$I_M(S)(\xi) = \begin{bmatrix} & \tilde{T}^*(\xi) & & \\ \tilde{T}(\xi) & & & \\ & & 0 & \\ & & & 0 \end{bmatrix}.$$

We can assume that S is symmetric (if not, replace S by $(S + S^*)/2$). Replace now D by $D+T$ and A by $A + S$. Note that $\operatorname{index}(D) = \operatorname{index}(D+T)$, since $T : H^1_c \to L^2_c$ is compact. The hypothesis of the theorem (in particular (3)) still hold for $D + T$ instead of D and with $A + S$ instead of A. So we can additionally assume that D is surjective or injective.

Construction of a positive cusp operator Q

For each $\xi \in \mathbb{R}$ we have $\sigma_1(\tilde{D}(\xi)) = \sigma_1(D)$, so $\tilde{D}(\xi)$ is elliptic as an operator on M and $\operatorname{index}(\tilde{D}(\xi)) = \operatorname{index}(D)$. If D is surjective or injective, then 0 does not belong to the spectrum of DD^* (respectively D^*D) so by continuity $\tilde{D}(\xi)$ will have the same property for small enough $|\xi|$. Thus there exists $\epsilon > 0$ such that the kernel and the cokernel of $\tilde{D}(\xi)$ have constant dimension (hence they vary smoothly) for all $|\xi| < \epsilon$. Choose a smooth real function ϕ supported in $[-\epsilon, \epsilon]$ such that $\phi(0) = 1$. By Lemma 1 and the choice of ϕ, the families $\phi(\xi)P_{\ker \tilde{D}(\xi)}$ and $\phi(\xi)P_{\ker \tilde{D}(\xi)}$ define suspended operators in $\Psi^{-\infty}_{\text{sus}}(M)$. Let $R \in \Psi_c^{-\infty,0}(X, \mathcal{E})$ be such that

$$I_M(R)(\xi) = \begin{bmatrix} \phi(\xi)P_{\ker \tilde{D}(\xi)} & & & \\ & \phi(\xi)P_{\operatorname{coker} \tilde{D}(\xi)} & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \tag{4}$$
$$\in \Psi^{-\infty}_{\text{sus}}(M, \mathcal{E}^+ \oplus \mathcal{E}^- \oplus V^+ \oplus V^-).$$

It follows that $I_M(A^2 + R^*R)(\xi)$ is invertible for all $\xi \in \mathbb{R}$, so the cusp operator $A^2 + R^*R$ is fully elliptic; this implies that it is Fredholm, and moreover its kernel is made of smooth sections vanishing rapidly towards ∂X. Let $P_{\ker(A^2+R^*R)}$ be the orthogonal projection on the finite-dimensional nullspace of $A^2 + R^*R$. Since moreover $A^2 + R^*R$ is clearly non-negative, it follows that $A^2 + R^*R + P_{\ker(A^2+R^*R)}$ is positive. Set

$$Q := (A^2 + R^*R + P_{\ker(A^2+R^*R)})^{1/2}$$

and let Q^λ be the complex powers of Q. Since $Q^2 - A^2 \in \Psi_c^{-\infty,0}(X,\mathcal{E})$ and A is self-adjoint, we deduce

$$[A, Q^\lambda] \in \Psi_c^{-\infty,0}(X,\mathcal{E}). \tag{5}$$

A non-commutative residue

Let $P(\lambda,z) \in \Psi_c^{-\lambda-1,-z-1}(X,\mathcal{E})$ be the analytic family of cusp operators

$$P(\lambda,z) := [x^z, A]Q^{-\lambda-1}.$$

From (2), $\mathrm{Tr}(P(\lambda,z))$ is holomorphic in $\{(\lambda,z)\in\mathbb{C}^2; \Re(\lambda) > \dim(X)-1, \Re(z) > 0\}$ and extends meromorphically to $\mathbb{C}^2$. Following the scheme of [17, Theorem 1], our proof of Theorem 3 will consist of computing in two different ways the complex number

$$N := \mathrm{Res}_{\lambda=0}\left(\mathrm{Tr}(P(\lambda,z))|_{z=0}\right),$$

i.e., N is the coefficient of $\lambda^{-1}z^0$ in the Laurent expansion of $\mathrm{Tr}(P(\lambda,z))$ around $(0,0)$. The idea is to evaluate at $z=0$ *before* and then *after* taking the residue at $\lambda=0$, noting that the final answer is independent of this order.

Vanishing of N

On one hand,

$$P(\lambda,z) = x^z[A, Q^{-\lambda-1}] + [A, Q^{-\lambda-1}x^z].$$

The meromorphic function $[A, Q^{-\lambda-1}x^z]$ is identically zero since it vanishes on the open set $\{(\lambda,z)\in\mathbb{C}^2; \Re(\lambda) > \dim(X)-1, \Re(z)>0\}$ by the trace property. By (5), the function $\mathrm{Tr}(x^z[A,Q^{-\lambda-1}])$ is regular in $\lambda\in\mathbb{C}$, so in particular the meromorphic function

$$z \mapsto \mathrm{Res}_{\lambda=0}\mathrm{Tr}(x^z[A,Q^{-\lambda-1}])$$

vanishes. We conclude that $N=0$.

Second computation of N

On the other hand, $P(\lambda,0)=0$ so

$$U(\lambda,z) := z^{-1}P(\lambda,z) \in \Psi_c^{-\lambda-1,-z-1}(X,\mathcal{E})$$

is an analytic family. Set $[\log x, A] := (z^{-1}[x^z,A])|_{z=0} \in \Psi_c^{0,1}(X,\mathcal{E})$. Then $U(\lambda,0) = [\log x, A]Q^{-\lambda-1}$. By multiplicativity of the indicial family,

$$I_M(x^{-1}U(\lambda,0)) = I_M(x^{-1}[\log x, A])I_M(Q^{-\lambda-1}).$$

By (3) and [8, Lemma 3.4], we see that $I_M(x^{-1}[\log x, A])$ is the 4×4 diagonal matrix

$$\begin{bmatrix} i & & & \\ & -i & & \\ & & i\xi(1+\xi^2+\Delta^+)^{-\frac12} & \\ & & & -i\xi(1+\xi^2+\Delta^-)^{-\frac12}\end{bmatrix}$$

and $I_M(Q^{-\lambda-1}) = I_M(A^2+R^*R)^{-\frac{\lambda+1}{2}}$. Also, using (4), we deduce that $I_M(A^2+R^*R)$ is the 4×4 diagonal matrix with entries

$$\begin{aligned}
a_{11} &= \xi^2 + \tilde{D}(\xi)^*\tilde{D}(\xi) + \phi^2(\xi)P_{\ker \tilde{D}(\xi)}\\
a_{22} &= \xi^2 + \tilde{D}(\xi)\tilde{D}(\xi)^* + \phi^2(\xi)P_{\mathrm{coker}\tilde{D}(\xi)}\\
a_{33} &= 1+\xi^2+\Delta^+\\
a_{44} &= 1+\xi^2+\Delta^-.
\end{aligned}$$

By (2),

$$\begin{aligned}
\mathrm{Tr}(P(\lambda,z))|_{z=0} &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}(I_M(x^{-1}(U(\lambda,0)))d\xi\\
&= \frac{i}{2\pi}\int_{\mathbb{R}} \Big(\mathrm{Tr}(\xi^2 + \tilde{D}(\xi)^*\tilde{D}(\xi) + \phi^2(\xi)P_{\ker \tilde{D}(\xi)})^{-\frac{\lambda+1}{2}}\\
&\quad - \mathrm{Tr}(\xi^2 + \tilde{D}(\xi)\tilde{D}(\xi)^* + \phi^2(\xi)P_{\mathrm{coker}\tilde{D}(\xi)})^{-\frac{\lambda+1}{2}}\\
&\quad + \xi\mathrm{Tr}(1+\xi^2+\Delta^+)^{-\frac{\lambda}{2}-1}\\
&\quad -\xi\mathrm{Tr}(1+\xi^2+\Delta^-)^{-\frac{\lambda}{2}-1}\Big)\, d\xi
\end{aligned}$$

The third and fourth terms in this sum are odd in ξ so their integral vanishes. For each fixed ξ we compute the trace of the first two terms by using orthonormal basis of $L^2_c(M,\mathcal{E}^+)$, $L^2_c(M,\mathcal{E}^+)$ given by eigensections of $\tilde{D}(\xi)^*\tilde{D}(\xi)$, respectively $\tilde{D}(\xi)\tilde{D}(\xi)^*$. The non-zero parts of the spectrum of $\tilde{D}(\xi)^*\tilde{D}(\xi)$ and $\tilde{D}(\xi)\tilde{D}(\xi)^*$ coincide, so what is left is

$$\int_{\mathbb{R}} \mathrm{index}(\tilde{D}(\xi))(\xi^2+\phi^2(\xi))^{-\frac{\lambda+1}{2}}d\xi.$$

The subtle point here is that the kernel and cokernel of $\tilde{D}(\xi)$ may have jumps when $|\xi|>\epsilon$, but our formula involves only the index, which is homotopy invariant and equals $\mathrm{index}(D)$ for all $\xi\in\mathbb{R}$. Thus the index comes out of the integral; the residue

$$\mathrm{Res}_{\lambda=0}\int_{\mathbb{R}}(\xi^2+\phi^2(\xi))^{-\frac{\lambda+1}{2}}d\xi$$

is independent of the compactly supported function ϕ and equals 2, so

$$0 = N = \mathrm{Res}_{\lambda=0}\mathrm{Tr}(P(\lambda,z))|_{z=0} = \frac{i}{\pi}\mathrm{index}(D). \qquad \square$$

By taking $\tilde{D}(\xi) := D$ with D differential, and $V^+ = V^- = 0$ we get the familiar form of cobordism invariance from e.g., [4], [7], [10], [18]. We gave a similar extension from the differential to the pseudodifferential case in [16] when computing the K-theory of the algebra $\Psi^0_{\mathrm{sus}}(M)$.

3. The K-theoretic characterization of cobordism invariance

We interpret now Theorem 2 in topological terms. Let $S^*_{\mathrm{sus}}(M)\to M$ be the sphere bundle inside $T^*_{\mathrm{sus}}M := T^*M\oplus\mathbb{R}$. The total space of $S^*_{\mathrm{sus}}(M)$ is the oriented boundary of ${}^cS^*X$. By fixing a product decomposition of X near M, we get non-canonical vector bundle isomorphisms making the diagram

$$\begin{tikzcd}
{}^cT^*X \arrow[r, "r"] \arrow[d, "\simeq"] & T^*_{\mathrm{sus}}M \arrow[d, "\simeq"]\\
T^*X \arrow[r, "r"] & T^*X|_M
\end{tikzcd}$$

commutative, so we can replace ${}^cS^*X$ with the more familiar space S^*X in all the topological considerations of this section.

The interior unit normal vector inclusion $\imath : M \to S^*_{\mathrm{sus}}(M)$ and the bundle projection $p : S^*_{\mathrm{sus}}(M) \to M$ induce a splitting

$$K^0(S^*_{\mathrm{sus}}(M)) = \ker(\imath) \oplus p^*(K^0(M)).$$

Let

$$r : K^0(S^*X) \to K^0(S^*_{\mathrm{sus}}(M))$$

be the map of restriction to the boundary, and

$$d : K^0(T^*M) \to K^0(S^*_{\mathrm{sus}}(M))/p^*(K^0(M))$$

the isomorphism defined as follows: if $(\mathcal{E}^+, \mathcal{E}^-, \sigma)$ is a triple defining a class in $K^0(T^*M)$ with $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ an isomorphism outside the open unit ball, then

$$d(\mathcal{E}^+, \mathcal{E}^-, \sigma) = \begin{cases} \mathcal{E}^+ & \text{on } S^*_{\mathrm{sus}}(M) \cap \{\xi \geqslant 0\} \\ \mathcal{E}^- & \text{on } S^*_{\mathrm{sus}}(M) \cap \{\xi \leqslant 0\} \end{cases}$$

with the two bundles identified via σ over $S^*_{\mathrm{sus}}(M) \cap \{\xi = 0\} = S^*M$. We can now reformulate Theorem 3 as follows:

Theorem 3. *Let X be a compact manifold with closed boundary M. Let $D \in \Psi(M, \mathcal{E}^+, \mathcal{E}^-)$ be an elliptic pseudodifferential operator and $[\sigma(D)] := (p^*\mathcal{E}^+, p^*\mathcal{E}^-, \sigma(D)) \in K^0(T^*M)$ its symbol class. Assume that*

$$d[\sigma(D)] \in p^*(K^0(M)) + r(K^0(S^*X)).$$

Then

$$\mathrm{index}(D) = 0.$$

Proof. The idea is to construct an operator A as in Theorem 2. We must first construct the vector bundles $V^\pm$, and then extend the principal symbol of (3) to an elliptic symbol in the interior of X.

We can assume that D is of order 1. Extend $\sigma(D)$ to a homomorphism $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ over $S^*_{\mathrm{sus}}(M)$. Let $\mathcal{F}^+ \to S^*_{\mathrm{sus}}(M)$ be the vector bundle defined as the span of the eigenvectors of positive eigenvalue of the symmetric automorphism of $p^*(\mathcal{E}^+ \oplus \mathcal{E}^-)$

$$a := \begin{bmatrix} \xi & \sigma^* \\ \sigma & -\xi \end{bmatrix} : p^*(\mathcal{E}^+ \oplus \mathcal{E}^-) \to p^*(\mathcal{E}^+ \oplus \mathcal{E}^-).$$

Lemma. *The K-theory class of the vector bundle $\mathcal{F}^+$ is $d[\sigma(D)]$.*

Proof. $\mathcal{F}^+$ is the image of the projector $\frac{1+a(a^2)^{-\frac{1}{2}}}{2}$ inside $p^*(\mathcal{E}^+ \oplus \mathcal{E}^-)$, or equivalently the image of the endomorphism $(a^2)^{\frac{1}{2}} + a$:

$$\begin{aligned} \mathcal{F}^+ = \{((\xi + (\xi^2 + \sigma^*\sigma)^{\frac{1}{2}})v, \sigma v); v \in \mathcal{E}^+\} \\ + \{(\sigma^* w, (-\xi + (\xi^2 + \sigma\sigma^*)^{\frac{1}{2}})w); w \in \mathcal{E}^-\}. \end{aligned}$$

Now $\xi + (\xi^2 + \sigma^*\sigma)^{\frac{1}{2}}$ is invertible when $\xi \geqslant 0$, and $-\xi + (\xi^2 + \sigma\sigma^*)^{\frac{1}{2}}$ is invertible when $\xi \leqslant 0$. Thus the projection from $\mathcal{F}^+$ on $p^*\mathcal{E}^+$, respectively on $p^*\mathcal{E}^-$, are isomorphisms for $\xi \geqslant 0$, respectively for $\xi \leqslant 0$. Over $\xi = 0$ these isomorphisms differ by $\sigma(\sigma^*\sigma)^{-\frac{1}{2}}$, which is homotopic to σ by varying the exponent from $-\frac{1}{2}$ to 0. □

The hypothesis says therefore that

$$[\mathcal{F}^+] \in p^*(K^0(M)) + r(K^0(S^*X)).$$

This means that there exist vector bundles $G^+ \to S^*X$, $V^+ \to M$ such that

$$\mathcal{F}^+ \oplus p^*V^+ = G^+|_{S^*_{\mathrm{sus}}(M)}. \tag{6}$$

Let $\mathcal{F}^- \subset p^*(\mathcal{E}^+ \oplus \mathcal{E}^-)$ be the negative eigenspace of a.

Lemma. *There exist vector bundles $G^- \to S^*X$, $V^- \to M$ such that*

$$\mathcal{F}^- \oplus p^*V^- = G^-|_{S^*_{\mathrm{sus}}(M)}$$

and moreover $\mathcal{E} := G^+ \oplus G^- \simeq \mathbb{C}^N$ for some $N \in \mathbb{N}$.

Proof. From (6) we get

$$\begin{aligned}[\mathcal{F}^-] &= [p^*(\mathcal{E}^+ \oplus \mathcal{E}^-)] - [\mathcal{F}^+]\\ &= p^*[\mathcal{E}^+ \oplus \mathcal{E}^- \oplus V^+] - r[G^+].\end{aligned}$$

Let G^0 be a complement (inside $\mathbb{C}^{N_0}$) of G^+, and V^0 a complement of $\mathcal{E}^+ \oplus \mathcal{E}^- \oplus V^+$ inside $\mathbb{C}^{N_1}$. Then

$$[\mathcal{F}^-] + p^*[V^0] + \mathbb{C}^{N_0} = \mathbb{C}^{N_1} + r[G^0]$$

which amounts to saying that there exists $N_2 \in \mathbb{N}$ with

$$\mathcal{F}^- \oplus p^*V^0 \oplus \mathbb{C}^{N_0+N_2} \simeq \mathbb{C}^{N_1+N_2} \oplus G^0|_{S^*_{\mathrm{sus}}(M)}.$$

We set $V^- := V^0 \oplus \mathbb{C}^{N_0+N_2}$, $G^- := \mathbb{C}^{N_1+N_2} \oplus G^0$. ☐

Define $\tilde{a}$ to be the automorphism of $\mathcal{E}$ over S^*X that equals 1 on G^+ and -1 on G^-. By construction, $\tilde{a}|_{S^*_{\mathrm{sus}}(M)}$ and the automorphism $\begin{bmatrix} a & & \\ & 1 & \\ & & -1 \end{bmatrix}$ (written in the decomposition $\mathcal{E}|_{S^*_{\mathrm{sus}}(M)} = (\mathcal{E}^+ \oplus \mathcal{E}^-) \oplus V^+ \oplus V^-$) have the same spaces of eigenvectors of positive, respectively negative eigenvalue, so they are homotopic inside self-adjoint automorphisms. Glue this homotopy smoothly to $\tilde{a}$ over $X \cup (M \times [-1, 0]) \simeq X$, and then pull back the result to a self-adjoint automorphism α of $\mathcal{E}$ over X. Thus

$$\alpha|_{S^*_{\mathrm{sus}}(M)} = \begin{bmatrix} a & & \\ & 1 & \\ & & -1 \end{bmatrix}. \tag{7}$$

We extend α to $T^*X \setminus 0$ by homogeneity 1.

As noted at the beginning of this section, we replace S^*X by ${}^cS^*X$. By (7) and the definition of a, $\alpha|_{S^*_{\mathrm{sus}}(M)}$ coincides with the principal symbol of the right-hand side of (3). Therefore there exists an elliptic cusp operator $A \in \Psi^1_c(X, \mathcal{E})$ with $\sigma_1(A) = \alpha$ and with indicial family given by the symmetric suspended operator (3). By replacing A with $(A + A^*)/2$ we can assume A to be symmetric. The hypothesis of Theorem 2 is fulfilled, so we conclude that $\mathrm{index}(D) = 0$. ☐

4. Variants of Theorem 3

4.1 Carvalho's theorem

Carvalho [5] obtained a slightly different statement of cobordism invariance (her result holds for non-compact manifolds as well). Namely, in the context of Theorem 3 she proved that $\mathrm{index}(D) = 0$ provided that $[\sigma(D)]$ lies in the image of the composite map

$$K^1(T^*X) \xrightarrow{r} K^1(T^*M \oplus \mathbb{R}) \xrightarrow{\beta^{-1}} K^0(T^*M)$$

defined by restriction and by Bott periodicity. Let us show that this statement is equivalent to Theorem 3. Consider the relative pairs $S^*X \hookrightarrow B^*X$, $S^*_{\mathrm{sus}}(M) \hookrightarrow B^*_{\mathrm{sus}}M$, the inclusion map between them and the induced boundary maps in the long exact sequences in K-theory. We claim that we get a commutative diagram

$$\begin{tikzcd} K^0(S^*X) \arrow[r, two heads] \arrow[d, "r"] & K^1(T^*X) \arrow[d, "r"] \\ K^0(S^*_{\mathrm{sus}}(M)) \arrow[r, two heads] \arrow[d, "q"] & K^1(T^*_{\mathrm{sus}}M) \\ K^0(S^*_{\mathrm{sus}}(M))/p^*K^0(M) \arrow[r, "d^{-1}"] & K^0(T^*M) \arrow[u, "\beta"] \end{tikzcd} \tag{8}$$

Indeed, the upper square commutes by naturality and the lower one by checking the definitions. Moreover, the existence of nonzero sections in $T^*X \to X$ and $T^*_{\mathrm{sus}}M \to M$ shows that the rows are surjective. Also β, d are isomorphisms, so $d[\sigma(D)]$ lies in the image of $q \circ r$ if and only if $[\sigma(D)]$ lies in the image of $\beta^{-1} \circ r$. Thus Theorem 3 is equivalent to Carvalho's statement applied to closed manifolds. Our formulation is marginally simpler because it does not involve the Bott isomorphism.

4.2 An indirect proof of Theorem 3

We mentioned in the introduction that Theorem 3 follows from the Atiyah-Singer formula:

$$\mathrm{index}(D) = \langle M, \mathrm{Td}(TM) \cup p_*\mathrm{ch}([\sigma(D)])\rangle,$$

where p_* denotes the Thom isomorphism induced by $p : T^*M \to M$. Indeed, the normal bundle to M in X is trivial so $\mathrm{Td}(TM) = r(\mathrm{Td}(TX))$. We can view T^*M as an open subset of $S^*X_{|M}$ via the central projection from the interior pole of each sphere; the pull-back via this inclusion map of $d[\sigma(D)])$ coincides modulo $p^*K^0(M)$ with $[\sigma(D)]$, in particular the push forward on M of $\mathrm{ch}(d[\sigma(D)])$ and of $\mathrm{ch}([\sigma(D)])$ are equal. So the hypothesis that $d[\sigma(D)]$ is the restriction of a class on S^*X modulo $p^*K^0(M)$ implies, by the functoriality of the Chern character, that $p_*\mathrm{ch}([\sigma(D)]) \in H^*(M)$ is the image of a cohomology class from X. Finally Stokes formula shows that $\mathrm{index}(D) = 0$.

References

1 M.F. Atiyah, V.K. Patodi and I.M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.

2 M. F. Atiyah and I. M. Singer, *The index of elliptic operators on compact manifolds,* Bull. Amer. Math. Soc. **69** (1963), 422–433.

3 M. F. Atiyah and I. M. Singer, *The index of elliptic operators. I,* Ann. of Math. **87** (1968), 484–530.

4 M. Braverman, *New proof of the cobordism invariance of the index,* Proc. Amer. Math. Soc. **130** (2002), 1095–1101.

5 C. Carvalho, *Pseudodifferential operators and applications to Index Theory on noncompact manifolds,* PhD thesis, Trinity College, University of Oxford (2003).

6 C. Carvalho, *A K-theory proof of the cobordism invariance of the index,* preprint (2004).

7 N. Higson, *A note on the cobordism invariance of the index,* Topology **30** (1991), 439–443.

8 R. Lauter and S. Moroianu, *The index of cusp operators on manifolds with corners,* Ann. Global Anal. Geom. **21** (2002), 31–49.

9 R. Lauter et S. Moroianu, *An index formula on manifolds with fibered cusp ends,* preprint math.DG/0212239 (2002).

10 M. Lesch, *Deficiency indices for symmetric Dirac operators on manifolds with conic singularities,* Topology **32** (1993), 611–623.

11 P. Loya, *Tempered operators and the heat kernel and complex powers of elliptic pseudodifferential operators,* Comm. Partial Differential Equations **26** (2001), 1253–1321.

12 R. B. Melrose, *The Atiyah-Patodi-Singer index theorem,* Research Notes in Mathematics **4**, A. K. Peters, Wellesley, MA (1993).

13 R. B. Melrose, *Pseudodifferential operators, corners and singular limits,* in Proc. Int. Congress of Mathematicians, Kyoto, Springer-Verlag Berlin - Heidelberg - New York (1990), 217–234.

14 R. B. Melrose, *The eta invariant and families of pseudodifferential operators,* Math. Res. Letters **2** (1995), 541–561.

15 R. B. Melrose and V. Nistor, *Homology of pseudodifferential operators I. Manifolds with boundary,* preprint funct-an/9606005.

16 S. Moroianu, *K-Theory of suspended pseudodifferential operators,* K-Theory 28 (2003), 167–181.

17 S. Moroianu, *Cusp geometry and the cobordism invariance of the index,* preprint math.DG/0305395, to appear in Adv. Math.

18 L. I. Nicolaescu, *On the cobordism invariance of the index of Dirac operators,* Proc. Amer. Math. Soc. **125** (1997), 2797–2801.

19 B. Vaillant, *Index- and spectral theory for manifolds with generalized fibered cusps,* Dissertation, Bonner Mathematische Schriften **344** (2001), Rheinische Friedrich-Wilhelms-Universität Bonn.

Sergiu Moroianu moroianu@alum.mit.edu
Institutul de Matematică al Academiei Române, P.O. Box 1-764, RO-014700 Bucharest, Romania
Current address: Université Paul Sabatier, UFR MIG, 118 route de Narbonne, 31062 Toulouse, France