Cusp geometry and the cobordism invariance of the index

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Abstract

The cobordism invariance of the index on closed manifolds is reproved using the calculus Ψ_c of cusp pseudodifferential operators on a manifold with boundary. More generally, on a compact manifold with corners, the existence of a symmetric cusp differential operator of order 1 and of Dirac type near the boundary implies that the sum of the indices of the induced operators on the hyperfaces is null.

Key words: Cusp pseudodifferential operators, noncommutative residues, zeta functions.

1 Introduction

Thom's discovery [21] of the cobordism invariance of the topological signature led Hirzebruch [10] to identify the signature of the intersection form of a closed oriented $4k$-dimensional manifold with the L-number constructed from the Pontryagin classes, in what was to become one of last century's most influential formulae:

$$\mathrm{sign}(M) = L(M).$$

Inspired by this result, Atiyah and Singer proposed in [1] an extension of the signature formula which gave the answer to the general index problem for

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elliptic operators on closed manifolds. Their program was carried out in [20]. The key ingredients of the proof were the use of pseudodifferential operators and the cobordism invariance of the index of twisted signature operators. Instead of explaining what this is, let us state a more general result, that we will later extend to manifolds with corners.

Theorem 1 *Let M be a closed manifold, $\mathcal{E}^{\pm}$ vector bundles over M and $D : \mathcal{C}^{\infty}(M, \mathcal{E}^{+}) \to \mathcal{C}^{\infty}(M, \mathcal{E}^{-})$ an elliptic differential operator of order 1. Assume that*

(1) M (not necessarily orientable) is the boundary of a compact manifold X; fix a Riemannian metric on X which is of product type in a product decomposition $M \times [0, \epsilon)$ of X near M;

(2) there exists a vector bundle $\mathcal{E} \to X$ such that $\mathcal{E}|_M = \mathcal{E}^{+} \oplus \mathcal{E}^{-}$; identify $\mathcal{E}$ over $M \times [0, \epsilon)$ with the pull-back of $\mathcal{E}^{+} \oplus \mathcal{E}^{-}$ from M, and fix a metric on $\mathcal{E}$ which is constant in $t \in [0, \epsilon)$ such that $\mathcal{E}^{+} \perp \mathcal{E}^{-}$;

(3) there exists a formally self-adjoint elliptic operator δ acting on $\mathcal{C}^{\infty}(X, \mathcal{E})$ which near M has the form

$$\delta = \begin{bmatrix} -i\frac{d}{dt} & D^{*} \\ D & i\frac{d}{dt} \end{bmatrix}. \tag{1}$$

Then $\operatorname{index}(D) = 0$.

Since the topological significance of $\ker D$, which played a key role for the signature problem, is lost for arbitrary elliptic operators, the proof from [20, Chapter XVII] of even a particular case of Theorem 1 had to rely on a fairly complicated analysis of boundary value problems. Atiyah and Singer found later [2] a purely K-theoretic proof of the index theorem, from which the cobordism invariance of twisted signatures follows. From a modern perspective, Theorem 1 is also a consequence of the following commutative diagram in analytic K-homology [9] (I am indebted to the referee for this remark):

$$\begin{array}{ccc} K_1(X, M) & \xrightarrow{\partial} & K_0(M) \\ \downarrow & & \downarrow{\scriptstyle \text{index}} \\ K_1(\mathrm{pt}, \mathrm{pt}) & \longrightarrow & K_0(\mathrm{pt}) \end{array} \tag{2}$$

An operator δ as in Theorem 1 defines an element in $K_1(X, M)$ with $\partial(\delta) = D \in K_0(M)$. On the other hand, $K_1(\mathrm{pt}, \mathrm{pt}) = 0$ so $\operatorname{index}(D) = 0$. Nevertheless, there is a great deal of work in either proving the Atiyah-Singer formula or in constructing analytic K-homology and proving the commutativity of (2). Thus it is legitimate to ask how deep the cobordism invariance of the index really is.

Our first result is a new proof of Theorem 1 by some clever manipulations with noncommutative residues inside the calculus of cusp pseudodifferential operators on X (arguably the simplest example of a pseudodifferential calculus on a manifold with boundary, constructed using the theory of boundary fibration structures of Melrose [14]). Note that several proofs of Theorem 1 have been obtained lately for Dirac operators (e.g., [3], [8], [12], [19]). A K-theoretic statement of the cobordism invariance of the index was proved recently by Carvalho [6,7] via the topological approach of [2].

The main result of the paper concerns the cobordism invariance problem on manifolds with corners. Let X be a compact manifold with corners and $\mathcal{F}_1, \dots, \mathcal{F}_k$ its boundary hyperfaces, possibly disconnected. We refer to [11] for an overview of cusp pseudodifferential operators on manifolds with corners. Let A be a symmetric cusp pseudodifferential operator on X. Under certain algebraic conditions which we call "being of Dirac type at the boundary", A induces cusp elliptic operators D_j on each hyperface $\mathcal{F}_j$. We assume that these operators are fully elliptic, which is equivalent to D_j being Fredholm on suitable cusp Sobolev spaces. Then, under the assumption that A is a first-order differential operator, we prove in Theorem 4 that $\sum_{j=1}^{k} \operatorname{index}(D_j^+) = 0$. The proof is inspired from the closed case; we look at a certain meromorphic function of zeta-type in several complex variables. A special Laurent coefficient of this function will give on one hand the sum of the indices of D_j, and on the other hand it will vanish.

An index formula for fully elliptic cusp operators on manifolds with corners was given in [11]. Inadvertently, we stated there the result only for scalar operators, however the formula applies *ad literam* to operators acting on sections of a vector bundle. The result from Section 4 is in a certain sense the odd-dimensional version of that formula. Unlike in the closed case, it seems difficult to obtain the cobordism invariance directly from the general index formula. Note however that for admissible Dirac operators, Theorem 4 can be deduced from results of Loya [13, Theorem 8.11], Bunke [5, Theorem 3.14], and also from a particular case of [11, Theorem 5.2], since in that case the index density is a characteristic form. We discuss this briefly at the end of Section 4.

From a different point of view, Melrose and Rochon [17] use a slightly modified version of the cusp algebra to study the index of families of operators on manifolds with boundary. It would be interesting to combine their approach with ours, to treat for instance the cobordism invariance of the families index.

In Sections 2 and 3 we will use the notation and some simple results from [11]; we refer the reader to [16] for a thorough treatment of the cusp algebra on manifolds with boundary. In the second part of the paper dealing with manifolds with corners we will rely heavily on [11]. Some familiarity with that paper must therefore be assumed.

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2 Review of Melrose's cusp algebra

Let X be a compact manifold with boundary M, not necessarily orientable, and $x : X \to \mathbb{R}_+$ a boundary-defining function (i.e., $M = \{x = 0\}$ and dx is never zero at $x = 0$). A *cusp vector field* on X is a smooth vector field V such that $dx(V) \in x^2\mathcal{C}^\infty(X)$ (we remind the reader that $\mathcal{C}^\infty(X)$ is defined as the space of restrictions to X of smooth functions on the double of X, or equivalently as the space of those smooth functions in the interior of X that admit Taylor series expansions at M). The space of cusp vector fields forms a Lie subalgebra ${}^c\mathcal{V}(X) \hookrightarrow \mathcal{V}(X)$, whose universal enveloping algebra is by definition the algebra $\mathcal{D}_c(X)$ of scalar cusp differential operators. Moreover ${}^c\mathcal{V}(X)$ is a finitely generated projective $\mathcal{C}^\infty(X)$-module (in a product decomposition $M \times [0, \epsilon) \hookrightarrow X$, a local basis is given by $\{x^2\frac{\partial}{\partial x}, \frac{\partial}{\partial y_j}\}$ where y_j are local coordinates on M). Thus by the Serre-Swan theorem there exists a vector bundle ${}^cTX \to X$ such that ${}^c\mathcal{V}(X) = \mathcal{C}^\infty(X, {}^cTX)$.

A cusp differential operator of positive order can never be elliptic at $x = 0$. Nevertheless, there exists a natural cusp principal symbol map surjecting onto the smooth polynomial functions on ${}^cT^*X$ of homogeneity k, $\sigma : \mathcal{D}_c^k(X) \to \mathcal{C}^\infty_{[k]}({}^cT^*X)$. A cusp operator will therefore be called elliptic if its principal symbol is invertible on ${}^cT^*X \setminus \{0\}$.

For any vector bundles $\mathcal{F}, \mathcal{G}$ over X let

$$\mathcal{D}_c(X, \mathcal{F}, \mathcal{G}) := \mathcal{D}_c(X) \otimes_{\mathcal{C}^\infty(X)} \mathrm{Hom}(\mathcal{F}, \mathcal{G}).$$

It is straightforward to extend the definition of σ to the bundle case.

2.1 Example

Assume that the hypothesis of Theorem 1 is fulfilled. The metric on X is a product metric near M,

$$g^X = dt^2 + g^M.$$

Extend δ to the manifold $\tilde{X} = M \times (-\infty, 0) \cup X$, obtained by attaching an infinite cylinder to X, by Eq. (1). Let $\psi : X^\circ \to \tilde{X}$ be any diffeomorphism extending

$$M \times (0, \epsilon) \ni (y, x) \mapsto (y, -\frac{1}{x}) \in \tilde{X}.$$

Then the pull-back of δ through ψ takes the form

$$A := \psi^* \delta = \begin{bmatrix} -ix^2 \frac{d}{dx} & D^* \\ D & ix^2 \frac{d}{dx} \end{bmatrix}$$

since $t = -\frac{1}{x}$ near $x = 0$. Thus, A is a cusp differential operator. Moreover, A is symmetric with respect to the cusp metric $\psi^* g^X$, which near M takes the form

$$g_c^X = \frac{dx^2}{x^4} + g^M.$$

The metric g_c^X is degenerate at $x = 0$, however it is non-degenerate as a cusp metric in the sense that it induces a Riemannian metric on the bundle ${}^cTX \to X$. The operator A is elliptic (in the cusp sense) and acts as an unbounded operator in $L_c^2(X, \mathcal{E})$, the space of square-integrable sections in $\mathcal{E} \to X$ with respect to the metric g_c^X.

2.2 The indicial family

This is a "boundary symbol" map, associating to any cusp operator $P \in \mathcal{D}_c(X, \mathcal{E}, \mathcal{F})$ a family of differential operators on M with one real polynomial parameter ξ as follows:

$$I_M(P)(\xi) = \left(e^{\frac{i\xi}{x}} P e^{-\frac{i\xi}{x}} \right) |_M$$

where restriction to M is justified by the mapping properties

$$\begin{aligned} P : \mathcal{C}^\infty(X, \mathcal{E}) &\to \mathcal{C}^\infty(X, \mathcal{F}) \\ P : x\mathcal{C}^\infty(X, \mathcal{E}) &\to x\mathcal{C}^\infty(X, \mathcal{F}) \end{aligned}$$

and by the isomorphism $\mathcal{C}^\infty(M) = \mathcal{C}^\infty(X)/x\mathcal{C}^\infty(X)$.

From the definition we see directly for the cusp operator A constructed in Example 2.1 that

$$I_M(A) = \begin{bmatrix} \xi & D^* \\ D & -\xi \end{bmatrix}. \tag{3}$$

Ellipticity does not make A Fredholm on $L^2_c(X, \mathcal{E})$, essentially because the Rellich lemma does not hold on non-compact domains. To apply a weighted form of the Rellich lemma we need an extra property, the invertibility of the indicial family $I_M(A)$ for all values of the parameter ξ; thus A is Fredholm precisely when D is invertible, see [11, Theorem 3.3]. Elliptic cusp operators with invertible indicial family are called fully elliptic.

2.3 Cusp pseudodifferential operators

By a micro-localization process one constructs [16] a calculus of pseudodifferential operators $\Psi^\lambda_c(X)$, $\lambda \in \mathbb{C}$, in which $\mathcal{D}_c(X)$ sits as the subalgebra of differential operators (the symbols used in the definition are classical of order λ). By composing with the multiplication operators x^z, $z \in \mathbb{C}$, we get a calculus with two indices

$$\Psi^{\lambda,z}_c(X, \mathcal{F}, \mathcal{G}) := x^{-z}\Psi^\lambda_c(X, \mathcal{F}, \mathcal{G})$$

such that $\Psi^{\lambda,z}_c(X, \mathcal{E}, \mathcal{F}) \subset \Psi^{\lambda',z'}_c(X, \mathcal{E}, \mathcal{F})$ if and only if $\lambda'-\lambda \in \mathbb{N}$ and $z'-z \in \mathbb{N}$ (since we work with classical symbols). Also,

$$\Psi^{\lambda,z}_c(X, \mathcal{G}, \mathcal{H}) \circ \Psi^{\lambda',z'}_c(X, \mathcal{F}, \mathcal{G}) \subset \Psi^{\lambda+\lambda',z+z'}_c(X, \mathcal{F}, \mathcal{H}).$$

By closure, cusp operators act on a scale of weighted Sobolev spaces $x^\alpha H^\beta_c$:

$$\Psi^{\lambda,z}_c(X, \mathcal{F}, \mathcal{G}) \times x^\alpha H^\beta_c(X, \mathcal{F}) \to x^{\alpha-\Re(z)} H^{\beta-\Re(\lambda)}_c(X, \mathcal{G}).$$

The principal symbol map and the indicial family extend to multiplicative maps on $\Psi_c(X)$. The indicial family takes values in the space $\Psi_{\mathrm{sus}}(M)$ of families of operators on M with one real parameter ξ, with joint symbolic behavior in ξ and in the cotangent variables of T^*M (1-suspended pseudodifferential operators in the terminology of [15]).

The following result gives a hint of what families of operators actually define suspended operators.

Lemma 2 *Let $z, w \in \mathbb{C} \cup \{-\infty\}$, $P \in \Psi^z(M)$ and $\phi \in \mathcal{C}^\infty(\mathbb{R})$. Then $\xi \mapsto \phi(\xi)P$ belongs to $\Psi^w_{\mathrm{sus}}(M)$ if and only if one of the following two conditions is fulfilled:*

(1) $z = -\infty$ and ϕ is a rapidly decreasing (i.e., Schwartz) function.
(2) P is a differential operator and ϕ is a polynomial.

In the first case $w = -\infty$, while in the second case $w = z + \deg(\phi)$.

2.4 Analytic families of cusp operators

Let $Q \in \Psi_c^{1,0}(X, \mathcal{E})$ be a positive fully elliptic cusp operator of order 1. Then the complex powers Q^λ form an analytic family of cusp operators of order λ (this is proved using Bucicovschi's method [4]).

Let $\mathbb{C}^2 \ni (\lambda, z) \mapsto P(\lambda, z) \in \Psi_c^{\lambda,z}(X, \mathcal{E})$ be an analytic family in two complex variables. Then $P(\lambda, z)$ is of trace class (on $L^2_c(M, \mathcal{E})$) for

$$\Re(\lambda) < -\dim(X), \quad \Re(z) < -1,$$

and $\mathrm{Tr}(P)$ is analytic there as a function of (λ, z). Moreover, $\mathrm{Tr}(P)$ extends to $\mathbb{C}^2$ meromorphically with at most simple poles in each variable at $\lambda \in \mathbb{N} - \dim(X)$, $z \in \mathbb{N} - 1$. By analogy with the Wodzicki residue, we can give a formula for the residue at $z = -1$ as a meromorphic function of λ (this is essentially [11, Proposition 4.5]).

Proposition 3 *Let $\mathbb{C}^2 \ni (\lambda, z) \mapsto P(\lambda, z) \in \Psi_c^{\lambda,z}(X, \mathcal{E})$ be an analytic family. Then*

$$\mathrm{Res}_{z=-1} \mathrm{Tr}(P(\lambda, z)) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{Tr}(I_M(x^{-1}P(\lambda, -1)))d\xi. \tag{4}$$

PROOF. The trace on the right-hand side is the trace on $L^2(M, \mathcal{E}_{|M})$. Both terms are meromorphic functions in $\lambda \in \mathbb{C}$. By unique continuation, it is thus enough to prove the identity for $\Re(\lambda) < -\dim(X)$. We write both traces as the integrals of the trace densities of the corresponding operators, i.e., the restriction of their distributional kernel to the diagonal.

For any vector bundle $V \to X$ we denote by $\Omega(V) \to X$ the associated density bundle. Let $F : \mathbb{C} \to \mathcal{C}^\infty(X, \Omega({}^cTX))$ be a holomorphic family of smooth cusp-densities. Then $x^2F(z) \in \mathcal{C}^\infty(X, \Omega(TX))$, and hence $z \mapsto \int_X x^{-z}F(z)$ is holomorphic for $\Re(z) < -1$; moreover, its residue at $z = -1$ is easily seen to equal $\int_M (x^2\partial_x \lrcorner F(-1))_{|M}$. We apply this fact to the trace density of $P(\lambda, z)$

multiplied with x^z. We view M as the intersection of the cusp diagonal with the cusp front face inside the cusp double space X_c^2 (see [11]). Recall from [11] or [16] that the cusp front face is the total space of a real line bundle, and M lives inside the zero section. The indicial family is obtained by restricting a Schwartz kernel to the front face, then Fourier transforming along the fibers. The result follows from the Fourier inversion formula

$$f(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) d\xi$$

applied in the fibers of the cusp front face over M.

3 Cobordism invariance on manifolds with boundary

The self-contained proof of the cobordism invariance of the index given below serves as a model for the general statement on manifolds with corners.

Proof of Theorem 1. We have seen in Example 2.1 that the hypothesis of Theorem 1 is equivalent to the existence of an elliptic symmetric cusp operator A satisfying (3).

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-negative Schwartz function with $\phi(0) = 1$. Let $P_{\ker D} \in \Psi^{-\infty}(M, \mathcal{E}^+)$, $P_{\operatorname{coker} D} \in \Psi^{-\infty}(M, \mathcal{E}^-)$ be the (finite-rank) orthogonal projections on the kernel and cokernel of D. These projections belong to $\Psi^{-\infty}(M)$ by elliptic regularity. By Lemma 2,

$$r(\xi) := \begin{bmatrix} \phi(\xi) P_{\ker D} & 0 \\ 0 & \phi(\xi) P_{\operatorname{coker} D} \end{bmatrix}$$

belongs to $\Psi_{\mathrm{sus}}^{-\infty}(M, \mathcal{E}^+ \oplus \mathcal{E}^-)$ and is non-negative, so it is the indicial family of a non-negative cusp operator $R \in \Psi_c^{-\infty,0}(X, \mathcal{E})$. By (3),

$$I_M(A^2 + R) = \begin{bmatrix} \xi^2 + D^*D + \phi(\xi) P_{\ker D} & 0 \\ 0 & \xi^2 + DD^* + \phi(\xi) P_{\operatorname{coker} D} \end{bmatrix} \tag{5}$$

so $A^2 + R$ is fully elliptic. It follows that $P_{\ker(A^2+R)} \in \Psi_c^{-\infty,-\infty}(X, \mathcal{E})$ (by elliptic regularity with respect to the two symbol structures) so $A^2 + R + P_{\ker(A^2+R)}$ is a positive cusp operator. Finally set

$$Q := (A^2 + R + P_{\ker(A^2+R)})^{1/2}$$

and let Q^λ be its complex powers. Note that $Q^2 - A^2 \in \Psi_c^{-\infty,0}(X,\mathcal{E})$ so

$$[A, Q^\lambda] \in \Psi_c^{-\infty,0}(X,\mathcal{E}). \tag{6}$$

Let $P(\lambda, z) \in \Psi_c^{-\lambda-1,-z-1}(X,\mathcal{E})$ be the analytic family of cusp operators

$$P(\lambda, z) := [x^z, A]Q^{-\lambda-1}.$$

From the discussion in Subsection 2.4, $\mathrm{Tr}(P(\lambda, z))$ is holomorphic in $\{(\lambda, z) \in \mathbb{C}^2; \Re(\lambda > \dim(X)-1, \Re(z) > 0\}$ and extends meromorphically to $\mathbb{C}^2$. We keep the notation $\mathrm{Tr}(P(\lambda, z))$ for this extension. Note that although $P(\lambda, 0) = 0$, there is no reason to expect the meromorphic extension $\mathrm{Tr}(P(\lambda, z))$ to vanish at $z = 0$; rather, $\mathrm{Tr}(P(\lambda, z))$ will be regular in z near $z = 0$. Our proof of Theorem 1 will consist of computing in two different ways the complex number

$$N := \mathrm{Res}_{\lambda=0} \left(\mathrm{Tr}(P(\lambda, z))|_{z=0}\right)$$

where $(\cdot)|_{z=0}$ denotes the regularized value in z at $z = 0$, which is a meromorphic function of λ. In other words, N is the coefficient of $\lambda^{-1}z^0$ in the Laurent expansion of $\mathrm{Tr}(P(\lambda, z))$ around $(0, 0)$. Evidently, we can also take the residue in λ *before* evaluating at $z = 0$; in that case, the output of the residue is a meromorphic function in z.

On one hand, we claim that

$$\mathrm{Tr}(P(\lambda, z)) = \mathrm{Tr}(x^z[A, Q^{-\lambda-1}])$$

for all $\lambda, z \in \mathbb{C}$. Since $[x^z, A]Q^{-\lambda-1} = x^z[A, Q^{-\lambda-1}] + [x^z Q^{-\lambda-1}, A]$, the claim is equivalent to showing that the meromorphic function

$$(z, \lambda) \mapsto \mathrm{Tr}([x^z Q^{-\lambda-1}, A])$$

vanishes identically. Indeed, for $\Re(\lambda) > \dim(X), \Re(z) > 1$ this vanishing holds by the trace property, and unique continuation proves the claim in general. Furthermore, $x^z[A, Q^{-\lambda-1}] \in \Psi_c^{-\infty,-z}(X,\mathcal{E})$ by (6) so in fact

$$(\lambda, z) \mapsto \mathrm{Tr}(x^z[A, Q^{-\lambda-1}]) = \mathrm{Tr}(P(\lambda, z))$$

is analytic in $\lambda \in \mathbb{C}$. In conclusion $\mathrm{Tr}(P(\lambda, z))$ is regular in λ at $\lambda = 0$, so

$$N = 0. \tag{7}$$

On the other hand, $P(\lambda, 0) = 0$ so

$$U(\lambda, z) := z^{-1}P(\lambda, z) \in \Psi_c^{-\lambda-1,-z-1}(X, \mathcal{E})$$

is an analytic family. Set $[\log x, A] := (z^{-1}[x^z, A])|_{z=0} \in \Psi_c^{0,1}(X, \mathcal{E})$. Then $U(\lambda, 0) = [\log x, A]Q^{-\lambda-1}$ and

$$\begin{aligned} I_M(x^{-1}U(\lambda,0)) &= I_M(x^{-1}[\log x, A])I_M(Q^{-\lambda-1}) \\ &= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} I_M(A^2 + R + P_{\ker(A^2+R)})^{-\frac{\lambda+1}{2}} \end{aligned}$$

where $I_M(A^2 + R + P_{\ker(A^2+R)})$ is given by (5) because $I_M(P_{\ker(A^2+R)}) = 0$. Using (4) we get

$$\begin{aligned} \operatorname{Tr}(P(\lambda, z))|_{z=0} &= \operatorname{Res}_{z=0} \operatorname{Tr}(z^{-1}P(\lambda, z)) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \operatorname{Tr}(I_M(x^{-1}(U(\lambda, 0)))d\xi \\ &= \frac{i}{2\pi} \int_{\mathbb{R}} \Big(\operatorname{Tr}(\xi^2 + D^*D + \phi(\xi)P_{\ker D})^{-\frac{\lambda+1}{2}} \\ &\quad - \operatorname{Tr}(\xi^2 + DD^* + \phi(\xi)P_{\operatorname{coker} D})^{-\frac{\lambda+1}{2}} \Big) d\xi \end{aligned} \tag{8}$$

For each fixed ξ we compute the trace using an orthonormal basis of $L^2(X, \mathcal{E}^{\pm})$ given by eigensections of D^*D, respectively DD^*. Clearly the contributions of nonzero eigenvalues cancel in (8) so we are left with

$$\operatorname{Tr}(P(\lambda, z))|_{z=0} = \operatorname{index}(D) \int_{\mathbb{R}} (\xi^2 + \phi(\xi))^{-\frac{\lambda+1}{2}} d\xi.$$

The reader will easily convince herself that the residue

$$\operatorname{Res}_{\lambda=0} \int_{\mathbb{R}} (\xi^2 + \phi(\xi))^{-\frac{\lambda+1}{2}} d\xi$$

is independent of the Schwartz function ϕ and equals 2. Thus

$$\begin{aligned} N &= \operatorname{Res}_{\lambda=0} \operatorname{Tr}(P(\lambda, z))|_{z=0} \\ &= \frac{i}{\pi} \operatorname{index}(D). \end{aligned}$$

Together with (7) this finishes the proof of Theorem 1.

4 Cobordism invariance on manifolds with corners

Let X be a manifold with corners in the sense of Melrose [14]. Let $\mathcal{M}_1(X)$ be the set of boundary hyperfaces, possibly disconnected, and for $H \in \mathcal{M}_1(X)$ fix x_H a defining function for H. We fix a product cusp metric g^X on the interior of X, which means iteratively that near each hyperface H, g^X takes the form

$$g^X = \frac{dx_H^2}{x_H^4} + g^H$$

for a product cusp metric on H. The algebra of cusp differential operators on X is simply the universal enveloping algebra of the Lie algebra of smooth vector fields on X of finite length with respect to g^X. The algebra $\Psi_c(X)$ of cusp pseudodifferential operators was described in [11] (see for instance [14] for the general ideas behind such constructions). In this section the reader is assumed to be familiar with [11]. Our main result is inspired from Theorem 1.

Theorem 4 *Let X be a compact manifold with corners and*

$$D_H : H_c^1(H, \mathcal{E}_H^+) \to L_c^2(H, \mathcal{E}_H^-)$$

a fully elliptic cusp differential operator of order 1 for each hyperface H of X. Assume that there exists a hermitian vector bundle $\mathcal{E} \to X$ with product metric near the corners and $A \in \Psi_c^1(X, \mathcal{E})$ a (cusp) elliptic symmetric differential operator, such that for each $H \in \mathcal{M}_1(X)$, $\mathcal{E}|_H \cong \mathcal{E}_H^+ \oplus \mathcal{E}_H^-$ and

$$I_H(A)(\xi_H) = \begin{bmatrix} \xi_H & D_H^* \\ D_H & -\xi_H \end{bmatrix}. \tag{9}$$

Then

$$\sum_{H \in \mathcal{M}_1(X)} \text{index}(D_H) = 0.$$

Remark 5 *The existence of A requires the following compatibility condition for D_H, D_G near $H \cap G$:*

$$\begin{aligned} I_G(D_H) &= i\xi_G + D_{HG} \\ I_H(D_G) &= i\xi_H - D_{HG} \end{aligned}$$

where D_{HG} is a symmetric invertible differential operator on $\mathcal{E}_H^+$ over the corner $H \cap G$, and $\mathcal{E}_H^+$, $\mathcal{E}_H^-$, $\mathcal{E}_G^+$, $\mathcal{E}_G^-$ are identified over $G \cap H$ by elementary linear *algebra. We say that A satisfying (9) is* of Dirac type near the boundary, *since the spin Dirac operator on a manifold with corners satisfies this condition.*

Proof of Theorem 4 For each $H \in \mathcal{M}_1(X)$, the operator $I_H(A)$ is fully elliptic (as a suspended cusp operator), however it is invertible if and only if D_H is invertible (this is seen easily by looking at the diagonal operator $I_H(A)^2$). We are interested exactly in the case when D_H has non-zero index, thus typically A is not fully elliptic. Nevertheless, D_H is Fredholm and its kernel is made of smooth sections vanishing rapidly to the boundary faces of H. Equivalently, the orthogonal projection $P_{\ker D_H}$ belongs to $\Psi_c^{-\infty,-\infty}(H, \mathcal{E}_H^+ \oplus \mathcal{E}_H^-)$.

Let ϕ a cut-off function with support in $[-\epsilon, \epsilon]$, $\phi \geq 0$, $\phi(0) = 1$. Then (see Lemma 2)

$$r_H(\xi_H) = \begin{bmatrix} \phi(\xi_H) P_{\ker D_H} & \\ & \phi(\xi_H) P_{\operatorname{coker} D_H} \end{bmatrix}$$

belongs to $\Psi_{c,\text{sus}}^{-\infty,-\infty}(H, \mathcal{E}^+ \oplus \mathcal{E}^-)$ and is non-negative. Clearly $I_H(A)^2 + r_H$ is invertible; let $R_H \in \Psi_c^{-\infty,0}(X)$ with $I_H(R_H) = r_H$, $I_G(R_H) = 0$ for $G \neq H$ (possible since $I_G(r_H) = 0$) and $R_H^* = R_H \geq 0$. Let $R := \sum_{H \in \mathcal{M}_1(X)} R_H$. Then $A^2 + R \geq 0$ is fully elliptic; by elliptic regularity, $P_{\ker(A^2+R)}$ belongs to $\Psi_c^{-\infty,-\infty}(X)$. Finally, we set

$$Q := (A^2 + R + P_{\ker(A^2+R)})^{1/2}.$$

The crucial property of the invertible operator Q is that its complex powers, like Q itself, commute with A modulo $\Psi_c^{-\infty,0}(X, \mathcal{E})$.

Look at the function

$$\mathbb{C}^{\mathcal{M}_1(X)} \times \mathbb{C} \ni (z, \lambda) \mapsto N(z, \lambda) := \operatorname{Tr}(x^z [A, Q^{-\lambda-1}]),$$

where $x^z := x_{H_1}^{z_{H_1}} \cdot \ldots \cdot x_{H_k}^{z_{H_k}}$. Here we have fixed an order on the set $\mathcal{M}_1(X) = \{H_1, \ldots, H_k\}$. By a general argument [11, Proposition 4.3], such a function can have at most simple poles in each of the complex variables, occurring at certain integers. But the family of operators involved is of order $-\infty$ with respect to the operator order (because of the commutativity modulo $\Psi_c^{-\infty,0}(X, \mathcal{E})$). Thus in fact there is no pole in λ at $\lambda = 0$, in particular

$$\operatorname{Res}_{\lambda=0} N(z, \lambda)_{z=0} = 0.$$

For $\Re(z_H) > -1, \Re(\lambda) > \dim(X)$ the trace property allows us to write

$$\begin{aligned} N(z,\lambda) &= \sum_{j=1}^{k} \mathrm{Tr}(x_{H_1}^{z_{H_1}} \cdot \ldots [A, x_{H_j}^{z_{H_j}}] \ldots \cdot x_{H_k}^{z_{H_k}} Q^{-\lambda-1}) \\ &=: \sum_{H \in \mathcal{M}_1(X)} N_H(z,\lambda). \end{aligned}$$

By unique continuation this identity holds for all z, λ. Each term $N_H(z,\lambda)$ of the right-hand side is a meromorphic function with at most simple poles in each variable. In fact, $N_H(z,\lambda)$ is regular in z_H at $z_H = 0$, since $[A, x_H^{z_H}]$ vanishes when $z_H = 0$. By [11, Proposition 4.5] (see also Proposition 3), $N_H(z,\lambda)_{|z_H=0}$ is given by

$$\frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{Tr}(x_{H_1}^{z_{H_1}} \cdot \ldots I_H(x_H^{-1}[\log x_{H_1}, A]) \ldots \cdot x_{H_k}^{z_{H_k}} I_H(Q)^{-\lambda-1}) d\xi_H. \tag{10}$$

By [11, Lemma 3.4],

$$I_H(x_H^{-1}[\log x_H, A]) = \frac{1}{i} \frac{\partial I_H(A)(\xi_H)}{\partial \xi_H}.$$

Now $I_H(Q)$ is a diagonal matrix, so the trace from formula (10) can be decomposed using the splitting of $\mathcal{E}_{|H}$. For the terms coming from $\mathcal{E}^{\pm}$, notice that the corresponding coefficient in $\frac{\partial I_H(A)(\xi_H)}{\partial \xi_H}$ has the pleasant property of being central, since it equals $\pm i$. Let

$$\begin{aligned} \widehat{x_H}^z &:= x^z / x_H^{z_H}, \\ T_H^+(\xi_H) &:= (D_H^* D_H + \xi_H^2 + \phi(\xi) P_{\ker D_H})^{-\frac{1}{2}}, \\ T_H^-(\xi_H) &:= (D_H D_H^* + \xi_H^2 + \phi(\xi) P_{\ker D_H^*})^{-\frac{1}{2}}. \end{aligned}$$

With this notation we get

$$N_H(z,\lambda)_{|z_H=0} = \frac{i}{2\pi} \int_{\mathbb{R}} \mathrm{Tr}(\widehat{x_H}^z (T_H^+(\xi_H)^{-\lambda-1} - T_H^-(\xi_H)^{-\lambda-1})) d\xi_H. \tag{11}$$

The trace functional and $\widehat{x_H}^z$ are independent of ξ_H, so we commute them out of the integral. We use now the identity

$$D_H T_H^+(\xi_H)^w = T_H^-(\xi_H)^w D_H$$

valid for every $w \in \mathbb{C}$, to decompose

$$T_H^-(\xi_H)^{-\lambda-1} = D_H T_H^+(\xi_H)^{-\lambda-1} T_H^+(0)^{-2} D_H^* + T_H^+(\xi_H)^{-\lambda-1} P_{\ker D_H^*}$$

in its components acting on $(\ker D_H^*)^\perp$, $\ker D_H^*$. Thus

$$\begin{aligned}\int_{\mathbb{R}} T_H^-(\xi_H)^{-\lambda-1} d\xi_H =& D_H T_H(0)^{-\lambda-2} D_H^* \int_{\mathbb{R}} (1+\xi^2)^{-\lambda-1} d\xi \\ &+P_{\ker D_H^*} \int_{\mathbb{R}} (1+\phi(\xi_H))^{-\frac{\lambda+1}{2}} d\xi_H. \end{aligned} \tag{12}$$

Similarly

$$T_H^+(\xi_H)^{-\lambda-1} = T_H^+(\xi_H)^{-\lambda-1} T_H^+(0)^{-2} D_H^* D_H + T_H^+(\xi_H)^{-\lambda-1} P_{\ker D_H^*}$$

so

$$\begin{aligned}\int_{\mathbb{R}} T_H^+(\xi_H)^{-\lambda-1} d\xi_H =& T_H(0)^{-\lambda-2} D_H^* D_H \int_{\mathbb{R}} (1+\xi^2)^{-\lambda-1} d\xi \\ &+P_{\ker D_H} \int_{\mathbb{R}} (1+\phi(\xi_H))^{-\frac{\lambda+1}{2}} d\xi_H. \end{aligned} \tag{13}$$

We are interested in the residue $\operatorname{Res}_{\lambda=0} N_H(z,\lambda)|_{z=0}$. Using (12), (13) we isolate in (11) the contribution of the projectors on the kernel and cokernel of D_H, and then evaluate at $\widehat{z_H} = 0$. Note that these projectors belong to the ideal $\Psi_c^{-\infty,-\infty}(H)$ so their contribution is regular in $\widehat{z_H}$. Now the trace of a projector equals the dimension of its image, while the residue at $\lambda = 0$ of $\int_{\mathbb{R}} (1+\phi(\xi_H))^{-\frac{\lambda+1}{2}} d\xi_H$ has been seen to be 2. Hence the contribution of the projector terms equals $\frac{i}{\pi}\operatorname{index}(D_H)$.

The function $\lambda \mapsto \int_{\mathbb{R}} (1+\xi^2)^{-\lambda-1} d\xi$ is regular at $\lambda = 0$ with value π. We still need to examine $\sum_{H\in\mathcal{M}_1(X)} \operatorname{Res}_{\lambda=0} L_H(0,\lambda)$, where

$$L_H(\widehat{z_H},\lambda) := \operatorname{Tr}(\widehat{x_H}^z[(D_H^* D_H + P_{\ker D_H})^{-\frac{\lambda}{2}-1} D_H^*, D_H]).$$

The residue in λ of L_H at $\widehat{z_H} = 0$ does not vanish directly, as one might hope at this point. We write as before (using the trace property for large real parts and then invoking unique continuation)

$$L_H(\widehat{z_H},\lambda) = \sum_{l=1}^{k-1} \operatorname{Tr}\Big(x_{G_1}^{z_{G_1}} \ldots [D_H, x_{G_l}^{z_{G_l}}] \ldots x_{G_{k-1}}^{z_{G_{k-1}}}$$

$$\begin{aligned}&\cdot(D_H^* D_H + P_{\ker D_H})^{-\frac{\lambda}{2}-1} D_H^*\Big)\\ =: &\sum_{G\in\mathcal{M}_1(X)\setminus\{H\}} L_{HG}(\widehat{z_H},\lambda).\end{aligned}$$

We see that L_{HG} is regular in z_G at $z_G = 0$ since it is the trace of an analytic family of operators which vanishes at $z_G = 0$. Moreover, we can write down the value $L_{HG}(\widehat{z_H},\lambda)|_{z_G=0}$ using Proposition 3 (or rather [11, Proposition 4.5], its analog for higher codimensions). By [11, Lemma 3.4] and from Remark 5,

$$I_G(x_G^{-1}[D_H, \log x_G]) = i\frac{\partial I_G(D_H)(\xi_G)}{\partial \xi_G} = -1.$$

Now $I_G(P_{\ker D_H}) = 0$ (by full ellipticity of D_H) while $I_G(D_H)(\xi_G) = i\xi_G + D_{HG}$. The term $i\xi_G$ contributes an odd integral in ξ_G to $L_{HG}|_{z_G=0}$, so

$$L_{HG}|_{z_G=0} = -\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}(\widehat{x_{HG}}^{z_{HG}}(D_{HG}^2+\xi_G^2)^{-\frac{\lambda}{2}-1}D_{HG})d\xi_G.$$

The argument is finished by Remark 5: indeed, modulo a conjugation, $D_{HG} = -D_{GH}$ so the above integrand appears again with opposite sign in $L_{GH}|_{z_H=0}$, once we replace the variables of integration ξ_G, ξ_H with a more neutral ξ. In other words, $\sum_{H,G\in\mathcal{M}_1(X)} L_{HG}(\widehat{z_H},\lambda)|_{\widehat{z_H}=0} = 0$, which together with our discussion on the projectors on the kernel of D_H shows that

$$\sum_{H\in\mathcal{M}_1(X)} \mathrm{Res}_{\lambda=0} N_H(z,\lambda)|_{z=0} = \sum_{H\in\mathcal{M}_1(X)} \mathrm{index}(D_H).$$

The left-hand side is just $\mathrm{Res}_{\lambda=0} N(z,\lambda)|_{z=0}$, which was seen to vanish.

Recall from [11] that the index of D_H can be written as the (regularized) integral on H of a density depending on the full symbol of D_H, plus contributions from each corner of H. In the case of differential operators of order 1 only the hyperfaces of H have a non-zero contribution, which is to be thought of as some sort of eta invariant. The eta invariant is sensitive to the orientation; in our case this means that $G \cap H$ contributes to the index of D_H and D_G the same quantity with opposite signs. Thus only the local index density detects whether our family of operators $\{D_H\}$ is cobordant to 0 or not. If we work with twisted Dirac operators, the local index density is given by a characteristic form with compact support away from the boundary of H. The existence of A as in Theorem 4 ensures that this characteristic form is the restriction of a characteristic form from X to the hyperfaces. Thus we deduce Theorem 4 in this case by Stokes formula. An index formula on a manifold with corners H was given in [5] for b-Dirac operators, in [13] for b-differential operators of order 1 and in [11] for cusp pseudodifferential operators. In this last paper the formula as stated covers scalar operators, but in reality it applies to operators acting on the sections of a vector bundle over H. This includes the case of Dirac operators if, surprisingly, the boundary of H is not empty. Indeed, in that case there exists a non-zero vector field on H which identifies, via the principal symbol map, any two bundles (e.g., the positive and negative spinor bundles) related by an elliptic operator.

5 A conjecture

We conjecture that Theorem 4 remains true for cusp pseudodifferential operators of order 1 of Dirac type near the boundary, in the sense of Remark 5. In this generality, our proof breaks down for instance when integrating with respect to ξ_H. For differential operators, we managed to show that the errors are concentrated at codimension 2 corners, and cancel each other. This seems not possible to do in the general case. One way to proceed would be to consider cusp operators of order $(1, 1)$, obtained by multiplying A with the inverse of the boundary defining functions. Then a power of such an operator of sufficiently small real part would be of trace class. Unfortunately, other complications arise, for instance the meromorphic extension of such a trace will have poles of higher order.

References

[1] M. F. Atiyah and I. M. Singer, *The index of elliptic operators on compact manifolds,* Bull. Amer. Math. Soc. **69** (1963), 422–433.

[2] M. F. Atiyah and I. M. Singer, *The index of elliptic operators. I,* Ann. of Math. **87** (1968), 484–530.

[3] M. Braverman, *New proof of the cobordism invariance of the index,* Proc. Amer. Math. Soc. **130** (2002), 1095–1101.

[4] B. Bucicovschi, *An extension of the work of V. Guillemin on complex powers and zeta functions of elliptic pseudodifferential operators,* Proc. Amer. Math. Soc. **127**, (1999), 3081–3090.

[5] U. Bunke, *Index theory, eta forms, and Deligne cohomology,* preprint (2002), math.DG/0201112.

[6] C. Carvalho, *Pseudodifferential operators and applications to Index Theory on noncompact manifolds,* PhD thesis, Trinity College, University of Oxford (2003).

[7] C. Carvalho, *A K-theory proof of the cobordism invariance of the index,* preprint (2004).

[8] N. Higson, *A note on the cobordism invariance of the index,* Topology **30** (1991), 439–443.

[9] N. Higson and J. Roe, *Analytic K-homology,* Oxford University Press (2000).

[10] F. Hirzebruch, *Neue Topologische Methoden in der Algebraischen Geometrie,* Springer-Verlag, Berlin–Göttingen–Heidelberg (1956).

[11] R. Lauter and S. Moroianu, *The index of cusp operators on manifolds with corners,* Ann. Global Anal. Geom. **21** (2002), 31–49.

[12] M. Lesch, *Deficiency indices for symmetric Dirac operators on manifolds with conic singularities,* Topology **32** (1993), 611–623.

[13] P. Loya, *Tempered operators and the heat kernel and complex powers of elliptic pseudodifferential operators,* Comm. Partial Differential Equations **26** (2001), 1253–1321.

[14] R. B. Melrose, *Pseudodifferential operators, corners and singular limits,* Proceedings of the International Congress of Mathematicians (Kyoto,1990), Math. Soc. Japan, Tokyo (1991), 217–234.

[15] R. B. Melrose, *The eta invariant and families of pseudodifferential operators,* Math. Res. Letters **2** (1995), 541–561.

[16] R. B. Melrose and V. Nistor, *Homology of pseudodifferential operators I. Manifolds with boundary,* to appear in Amer. Math. J., funct-an/9606005.

[17] R. B. Melrose and F. Rochon, *Families Index for Pseudodifferential Operators on Manifolds with Boundary,* Int. Math. Res. Not. **2004** nr. 22, 1115–1141.

[18] S. Moroianu, *K-Theory of suspended pseudodifferential operators,* K-Theory 28 (2003), 167–181.

[19] L. I. Nicolaescu, *On the cobordism invariance of the index of Dirac operators,* Proc. Amer. Math. Soc. **125** (1997), 2797–2801.

[20] R. S. Palais, *Seminar on the Atiyah-Singer index theorem,* Annals of Mathematics Studies **57**, Princeton University Press (1965).

[21] R. Thom, *Espaces fibrés en sphères et carrés de Steenrod,* Ann. Sci. Ecole Norm. Sup. **69** (1952), 109–182.