SCIENTIFIC REPORT ON THE PROJECT PN-II-RU-TE-2011-3-0053 "QUANTUM INVARIANTS IN HYPERBOLIC GEOMETRY"

SERGIU MOROIANU (PROJECT DIRECTOR)

1. Details on the paper "Bergman and Calderón projectors for Dirac operators"

For a Dirac operator $D_{\bar{g}}$ over a spin compact Riemannian manifold with boundary $(\overline{X}, \bar{g})$, we give a natural construction of the Calderón projector and of the associated Bergman projector on the space of harmonic spinors on $\overline{X}$, and we analyze their Schwartz kernels. Our approach is based on the conformal covariance of $D_{\bar{g}}$ and the analysis of the complete conformal metric $g = \bar{g}/\rho^2$ where ρ is a smooth function on $\overline{X}$ which is equal to the distance to the boundary near $\partial\overline{X}$. We then show that $\frac{1}{2}(\mathrm{Id} + \widetilde{S}(0))$ is the orthogonal Calderón projector, where $\widetilde{S}(\lambda)$ is the holomorphic family in $\{\Re(\lambda) \geq 0\}$ of normalized scattering operators constructed in [22], which are classical pseudo-differential of order 2λ. Finally we construct natural conformally covariant odd powers of the Dirac operator on any compact spin manifold.

2. Details on the paper "Chern-Simons line bundle on Teichmüller space"

In [11], S.S. Chern and J. Simons defined secondary characteristic classes of connections on principal bundles, arising from Chern-Weil theory. Their work has been extensively developed to what is now called Chern-Simons theory, with many applications in geometry and topology, but also in theoretical physics. For a Riemannian oriented 3-manifold X, the Chern-Simons invariant $\mathrm{CS}(\omega, S)$ of the Levi-Civita connection form ω in an orthonormal frame S is given by the integral of the 3-form on X

$$\frac{1}{16\pi^2}\mathrm{Tr}(\omega \wedge d\omega + \tfrac{2}{3}\omega \wedge \omega \wedge \omega).$$

On closed 3-manifolds, the invariant $\mathrm{CS}(\omega)$ is independent of S up to integers. By the Atiyah-Patodi-Singer theorem for the signature operator, the Chern-Simons invariant of the Levi-Civita connection is related to the eta invariant by the identity $3\eta \equiv 2\mathrm{CS}$ modulo $\mathbb{Z}$ (see for instance [45]).

The theory has been extended to SU(2) flat connections on compact 3-manifolds with boundary by Ramadas-Singer-Weitsman [39], in which case $\mathrm{CS}(\omega)$ does depend on the boundary value of the section S. The Chern-Simons invariant $e^{2\pi i \mathrm{CS}(\cdot)}$ can be viewed as a section of a complex line bundle (with a Hermitian structure) over the moduli space of flat SU(2) connections on the boundary surface. They proved that this bundle is isomorphic

Date: November 27, 2014.

S. M. was partially supported by the CNCS project PN-II-RU-TE-2011-3-0053.

to the determinant line bundle introduced by Quillen [38]. Some more systematic studies and extensions of the Chern-Simons bundle have been developed by Freed [19] and Kirk-Klassen [29]. One contribution of our present work is to give an explicit isomorphism between these Hermitian holomorphic line bundles in the Schottky setting.

An interesting field of applications of Chern-Simons theory is for hyperbolic 3-manifolds $X = \Gamma\backslash\mathbb{H}^3$, which possess a natural flat connection θ over a principal $\mathrm{PSL}_2(\mathbb{C})$-bundle. For closed manifolds, Yoshida [45] defined the $\mathrm{PSL}_2(\mathbb{C})$-Chern-Simons invariant as above by

$$\mathrm{CS}(\theta) = -\tfrac{1}{16\pi^2}\int_X S^*\left(\mathrm{Tr}(\theta\wedge d\theta + \tfrac{2}{3}\theta\wedge\theta\wedge\theta)\right)$$

where $S : X \to P$ are particular sections coming from the frame bundle over X. This is a complex number with imaginary part $-\frac{1}{2\pi^2}\mathrm{Vol}(X)$, and real part equal to the Chern-Simons invariant of the Levi-Civita connection on the frame bundle. Up to the contribution of a link in X, the function $F := \exp(\frac{2}{\pi}\mathrm{Vol}(M) + 4\pi i\mathrm{CS}(M))$ extends to a holomorphic function on a natural deformation space containing closed hyperbolic manifolds as a discrete set.

Our setting in this paper is that of 3-dimensional geometrically finite hyperbolic manifolds X without rank-1 cusps, in particular *convex co-compact hyperbolic* manifolds, which are conformally compactifiable to a smooth manifold with boundary. Typical examples are quotients of $\mathbb{H}^3$ by quasi-Fuchsian or Schottky groups. The ends of X are either funnels or rank-2 cusps. The funnels have a conformal boundary, which is a disjoint union of compact Riemann surfaces forming the *conformal boundary* M of X. The deformation space of X is essentially the deformation space of its conformal boundary, i.e. Teichmüller space. Before defining a Chern-Simons invariant, it is natural to ask about a replacement of the volume in this case. For Einstein conformally compact manifolds, the notion of *renormalized volume* $\mathrm{Vol}_R(X)$ has been introduced by Henningson-Skenderis [25] in the physics literature and by Graham [21] in the mathematical literature. In the particular setting of hyperbolic 3-manifolds, this has been studied by Krasnov [31] and extended by Takhtajan-Teo [41], in relation with earlier work of Takhtajan-Zograf [43], to show that Vol_R is a Kähler potential for the Weil-Petersson metric in Schottky and quasi-Fuchsian settings. Krasnov and Schlenker [37] gave a more geometric proof of this, using the Schläfli formula on convex co-compact hyperbolic 3-manifolds to compute the variation of Vol_R in the deformation space

Before we introduce the Chern-Simons invariant in our setting, let us first recall the definition of Vol_R used by Krasnov-Schlenker [37]. A *hyperbolic funnel* is some collar $(0,\epsilon)_x \times M$ equipped with a metric

$$(1)\quad g = \frac{dx^2 + h(x)}{x^2}, \quad h(x) \in \mathcal{C}^\infty(M, S^2_+T^*M), \quad h(x) = h_0\left((\mathrm{Id} + \tfrac{x^2}{2}A)\cdot, (\mathrm{Id} + \tfrac{x^2}{2}A)\cdot\right)$$

where M is a Riemann surface of genus ≥ 2 with a hyperbolic metric h_0, A is an endomorphism of TM satisfying $\mathrm{div}_{h_0}A = 0$, and $\mathrm{Tr}(A) = -\frac{1}{2}\mathrm{scal}_{h_0}$. The metric g on the funnel is of constant sectional curvature -1, and every end of a convex co-compact hyperbolic manifold X is isometric to such a hyperbolic funnel, see [18, 37]. A couple (h_0, A_0) can be considered as an element of $T^*_{h_0}\mathcal{T}$, if $A_0 = A - \frac{1}{2}\mathrm{tr}(A)\mathrm{Id}$ is the trace-free part of the divergence-free tensor A. We therefore identify the cotangent bundle $T^*\mathcal{T}$ of $\mathcal{T}$ with the set of hyperbolic funnels modulo the action of the group $\mathcal{D}_0(M)$, acting trivially in the x variable. Let x be any smooth positive function on X which extends the function x defined in each funnel by (1), and is equal to 1 in each cusp end. The renormalized volume of (X, g) is defined by

$$\mathrm{Vol}_R(X) := \mathrm{FP}_{\epsilon\to 0}\int_{x>\epsilon} d\mathrm{vol}_g$$

where FP means finite-part (i.e. the coefficient of ϵ^0 in the asymptotic expansion as $\epsilon \to 0$).

The tangent bundle to any 3-manifold is trivial. If ω is the so(3)-valued Levi-Civita connection 1-form on X in an oriented orthonormal frame $S = (S_1, S_2, S_3)$, we define

$$\mathrm{CS}(g, S) := -\tfrac{1}{16\pi^2}\mathrm{FP}_{\epsilon\to 0}\int_{x>\epsilon} \mathrm{Tr}(\omega\wedge d\omega + \tfrac{2}{3}\omega\wedge\omega\wedge\omega). \tag{2}$$

We ask that S be even to the first order at $\{x = 0\}$ and also that, in each cusp end, S be parallel in the direction of the vector field pointing towards to cusp point. Equipped with the conformal metric $\hat{g} := x^2 g$, the manifold X extends to a smooth Riemannian manifold $\overline{X} = X \cup M$ with boundary M. The Chern-Simons invariant $\mathrm{CS}(\hat{g}, \hat{S})$ is therefore well defined if $\hat{S} = x^{-1}S$ is an orthonormal frame for $\hat{g}$. We define the $\mathrm{PSL}_2(\mathbb{C})$ Chern-Simons invariant $\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(g, S)$ on (X, g) by the renormalized integral (2) where we replace ω by the complex-valued connection form $\theta := \omega + iT$; here T is the so(3)-valued 1-form defined by $T_{ij}(V) := g(V \times S_j, S_i)$ and $\times$ is the vector product with respect to the metric g. There exists a natural flat connection on a $\mathrm{PSL}_2(\mathbb{C})$ principal bundle $F^{\mathbb{C}}(X)$ over X (which can be seen as a complexified frame bundle), with $\mathrm{sl}_2(\mathbb{C})$-valued connection 1-form Θ, and we show that $\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(g, S)$ also equals the renormalized integral of the pull-back of the Chern-Simons form $-\frac{1}{4\pi^2}\mathrm{Tr}(\Theta\wedge d\Theta + \frac{2}{3}\Theta^3)$ of the flat connection Θ. We first show

Proposition 2.1. *On a geometrically finite hyperbolic* 3*-manifold* (X, g) *without rank-*1 *cusps, one has* $\mathrm{CS}(g, S) = \mathrm{CS}(\hat{g}, \hat{S})$*, and*

$$\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(g, S) = -\tfrac{i}{2\pi^2}\mathrm{Vol}_R(X) + \tfrac{i}{4\pi}\chi(M) + \mathrm{CS}(g, S) \tag{3}$$

where $\chi(M)$ *is the Euler characteristic of the conformal boundary* M*.*

The relation between $\mathrm{CS}(g, S)$ and $\mathrm{CS}(\hat{g}, \hat{S})$ comes rather easily from the conformal change formula in the Chern-Simons form (the boundary term turns out to not contribute), while (3) is a generalization of a formula in Yoshida [45], but we give an independent easy proof. Similar identities to (3) can be found in the physics literature (see for instance [32]).

Like the function F of Yoshida, it is natural to consider the variation of $\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(g, S)$ in the set of convex co-compact hyperbolic 3-manifolds, especially since, in contrast with the finite volume case, there is a finite dimensional deformation space of smooth hyperbolic 3-manifolds, which essentially coincides with the Teichmüller space of their conformal boundaries. One of the problems, related to the work of Ramadas-Singer-Weitsman [39] is that $e^{2\pi i\mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}(g,S)}$ depends on the choice of the frame S, since X is not closed. This leads us to define a complex line bundle $\mathcal{L}$ over Teichmüller space $\mathcal{T}$ of Riemann surfaces of a fixed genus, in which $e^{2\pi i \mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}}$ and $e^{2\pi i \mathrm{CS}}$ are sections.

Let $\mathcal{T}$ be the Teichmüller space of a (not necessarily connected) oriented Riemann surface M of genus $\mathbf{g} = (g_1, \dots, g_N)$, $g_j \geq 2$, defined as the space of hyperbolic metrics on M modulo the group $\mathcal{D}_0(M)$ of diffeomorphisms isotopic to the identity. This is a complex simply connected manifold of complex dimension $3|\mathbf{g}|-3$, equipped with a natural Kähler metric called the Weil-Petersson metric. The *mapping class group* Mod of isotopy classes of orientation preserving diffeomorphisms of M acts properly discontinuously on $\mathcal{T}$. Let (X, g) be a geometrically finite hyperbolic 3-manifold without cusp of rank 1, with conformal boundary M. By Theorem 3.1 of [34], there is a smooth map Φ from $\mathcal{T}$ to the set of geometrically finite hyperbolic metrics on X (up to diffeomorphisms of X homotopic to identity) such that the conformal boundary of $\Phi(h)$ is (M, h) for any $h \in \mathcal{T}$. The subgroup Mod_X of Mod consisting of elements which extend to diffeomorphisms on $\overline{X}$ homotopic to the identity acts freely, properly discontinuously on $\mathcal{T}$ and the quotient is a complex manifold of dimension $3|\mathbf{g}| - 3$. The map Φ is invariant under the action of Mod_X and the deformation space $\mathcal{T}_X$ of X is identified with a quotient of the Teichmüller space $\mathcal{T}_X = \mathcal{T}/\mathrm{Mod}_X$, see [34, Th. 3.1] .

Theorem 2.2. *Let (X, g) be a geometrically finite hyperbolic 3-manifold without rank-1 cusp, and with conformal boundary M. There exists a holomorphic Hermitian line bundle $\mathcal{L}$ over $\mathcal{T}$ equipped with a Hermitian connection $\nabla^{\mathcal{L}}$, with curvature given by $\frac{i}{8\pi}$ times the Weil-Petersson symplectic form ω_{WP} on $\mathcal{T}$. The bundle $\mathcal{L}$ with its connection descend to $\mathcal{T}_X$ and if $g_h = \Phi(h)$ is the geometrically finite hyperbolic metric with conformal boundary $h \in \mathcal{T}$, then $h \to e^{2\pi i \mathrm{CS}(g_h, \cdot)}$ is a global section of $\mathcal{L}$.*

The line bundle is defined using the cocycle which appears in the Chern-Simons action under gauge transformations. We remark that the computation of the curvature of $\mathcal{L}$ reduces to the computation of the curvature of the vertical tangent bundle in a fibration related to the universal Teichmüller curve over $\mathcal{T}$, and we show that the fiberwise integral of the first Pontrjagin form of this bundle is given by the Weil-Petersson form, which is similar to a result of Wolpert [44]. An analogous line bundle, but in a more general setting, has been recently studied by Bunke [10].

Since funnels can be identified to elements in $T^*\mathcal{T}$, the map Φ described above induces a section σ of the bundle $T^*\mathcal{T}$ (which descends to $T^*\mathcal{T}_X$) by assigning to $h \in \mathcal{T}$ the funnels of $\Phi(h)$. The image of σ
$$\mathcal{H} := \{\sigma(h) \in T^*\mathcal{T}_X, h \in \mathcal{T}_X\}$$
identifies the set of geometrically finite hyperbolic metrics on X as a graph in $T^*\mathcal{T}_X$.

Let us still denote by $\mathcal{L}$ the Chern-Simons line bundle pulled-back to $T^*\mathcal{T}$ by the projection $\pi_{\mathcal{T}} : T^*\mathcal{T} \to \mathcal{T}$, and define a modified connection
$$\nabla^{\mu} := \nabla^{\mathcal{L}} + \tfrac{2}{\pi}\mu^{1,0} \tag{4}$$
on $\mathcal{L}$ over $T^*\mathcal{T}$, where $\mu^{1,0}$ is the $(1,0)$ part of the Liouville 1-form μ on $T^*\mathcal{T}$. As before, the connection descends to $T^*\mathcal{T}_X$, and notice that it is not Hermitian (since $\mu^{1,0}$ is not purely imaginary) but ∇^{μ} and $\nabla^{\mathcal{L}}$ induce the same holomorphic structure on $\mathcal{L}$.

By Theorem 2.2 and Proposition 2.1, $e^{2\pi i \mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}}$ is a section of $\mathcal{L}$ on $\mathcal{T}_X$, its pull-back by $\pi_{\mathcal{T}}$ also gives a section of $\mathcal{L}$ on $\mathcal{H}$, which we still denote $e^{2\pi i \mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}}$.

Theorem 2.3. *Let $V \in T\mathcal{H}$ be a vector field tangent to $\mathcal{H}$, then $\nabla^{\mu}_V e^{2\pi i \mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}} = 0$, i.e. ∇^{μ} is flat on $\mathcal{H} \subset T^*\mathcal{T}_X$.*

The curvature of ∇^{μ} vanishes on $\mathcal{H}$ by Theorem 2.3 while the curvature of $\nabla^{\mathcal{L}}$ is $\frac{i}{8\pi}\omega_{\mathrm{WP}}$ (by Theorem 2.2). By considering the real and imaginary parts of these curvature identities, we obtain as a direct corollary :

Corollary 2.4. *The manifold $\mathcal{H}$ is Lagrangian in $T^*\mathcal{T}_X$ for the Liouville symplectic form μ and $d(\mathrm{Vol}_R \circ \sigma) = -\frac{1}{4}\mu$ on $\mathcal{H}$. The renormalized volume is a Kähler potential for Weil-Petersson metric on $\mathcal{T}_X$:*

$$\bar{\partial}\partial(\mathrm{Vol}_R \circ \sigma) = \frac{i}{16}\omega_{\mathrm{WP}}.$$

Our final result relates the Chern-Simons line bundle $\mathcal{L}$ to the Quillen determinant line bundle $\det \partial$ of ∂ on functions in the particular case of Schottky hyperbolic manifolds. If M is a connected surface of genus $\mathbf{g} \geq 2$, one can realize any complex structure on M as a quotient of an open set $\Omega_\Gamma \subset \mathbb{C}$ by a Schottky group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ and using a marking $\alpha_1, \ldots, \alpha_{\mathbf{g}}$ of $\pi_1(M)$ and a certain normalization, there is complex manifold $\mathfrak{S}$, called the Schottky space, of such groups. This is isomorphic to $\mathcal{T}_X$, where $X := \Gamma\backslash\mathbb{H}^3$ is the solid torus bounding M in which the curves α_j are contractible. The Chern-Simons line bundle $\mathcal{L}$ can then be defined on $\mathfrak{S}$. The Quillen determinant bundle $\det \partial$ is equipped with its Quillen metric and a natural holomorphic structure induced by $\mathfrak{S}$, therefore inducing a Hermitian connection compatible with the holomorphic structure. Moreover, there is a canonical section of $\det \partial = \Lambda^{\mathbf{g}}(\mathrm{coker}\, \partial)$ given by $\varphi := \varphi_1 \wedge \cdots \wedge \varphi_{\mathbf{g}}$ where φ_j are holomorphic 1-forms on M normalized by the marking through the requirement $\int_{\alpha_j} \varphi_k = \delta_{jk}$. Using a formula of Zograf [46, 47], we show

Theorem 2.5. *There is an explicit isometric isomorphism of holomorphic Hermitian line bundles between the inverse $\mathcal{L}^{-1}$ of the Chern-Simons line bundle and the 6-th power $(\det \partial)^{\otimes 6}$ of the determinant line bundle $\det \partial$, given by*

$$(F\varphi)^{\otimes 6} \mapsto e^{-2\pi i \mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}}.$$

Here φ is the canonical section of $\det \partial$ defined above, $c_{\mathbf{g}}$ is a constant, and F is a holomorphic function on $\mathfrak{S}$ which is given, on the open set where the product converges absolutely, by

$$F(\Gamma) = c_{\mathbf{g}} \prod_{\{\gamma\}} \prod_{m=0}^{\infty} (1 - q_\gamma^{1+m}),$$

where q_γ is the multiplier of $\gamma \in \Gamma$, $\{\gamma\}$ runs over all distinct primitive conjugacy classes in $\Gamma \in \mathfrak{S}$ except the identity.

Novelties and perspectives. Our main contribution in this work is to introduce the Chern-Simons theory and its line bundle over Teichmüller space in relation with Kleinian groups. The strength of this construction appears through a variety of applications to Teichmüller theory in essentially the most general setting, all at once and self-contained. For example, the property of the renormalized volume of being a Kähler potential for the Weil-Petersson metric, previously known in the particular cases of Schottky and quasi-Fuchsian groups [31, 42, 43, 37], follows directly from our Chern-Simons approach for all geometrically finite Kleinian groups without cusps of rank 1 (for instance, the proof in [37] is based on an explicit computation at the Fuchsian locus and does not seem to be extendable to general groups). In fact, finding Kähler potentials for the Weil-Petersson metric starting from a general Kleinian cobordism is not only a generalisation of the quasi-Fuchsian and Schottky cases. Indeed, the Chern-Simons bundle $\mathcal{L}$ is a "prequantum bundle" and together with the canonical holomorphic sections $e^{2\pi i \mathrm{CS}^{\mathrm{PSL}_2(\mathbb{C})}}$ corresponding to each hyperbolic cobordism, it could be used for a geometric definition of a Topological Quantum Field Theory through the quantization of Teichmüller space. We shall pursue this question elsewhere.

The existence of a non-explicit isomorphism between the Chern-Simons bundle on the (compact) moduli space of SU(2) flat connections and the determinant line bundle was discovered in [39]. In contrast, in our non-compact $\mathrm{PSL}_2(\mathbb{C})$ setting we find an *explicit* isomorphism, involving a formula of Zograf on Schottky space, which as far as we know is the first of its kind; we expect to generalize this to all convex co-compact groups.

More generally, we expect the results of this paper to extend to all geometrically finite hyperbolic 3-manifolds. Several technical difficulties appear when we perform our analysis to cusps of rank 1, this will be carried out elsewhere.

3. Details on the paper "Currents on locally conformally Kähler manifolds"

A locally conformally Kähler manifold (LCK for short) is a Hermitian manifold (M, J, g) for which the fundamental two-form $\omega(X, Y) = g(JX, Y)$ satisfies

$$d\omega = \theta \wedge \omega, \qquad d\theta = 0 \tag{5}$$

for some one-form θ called the Lee form.

There are many examples of compact LCK and non-Kähler manifolds, among them the Hopf manifolds.

As $d\theta = 0$, the twisted differential $d_\theta := d - \theta\wedge$ defines a twisted cohomology which is the Morse-Novikov cohomology of X. The LCK condition simply means that the fundamental form of (X, J, g) is d_θ-closed.

The aim of this paper is to obtain an analogue of the intrinsic characterization in [HL] for Kähler manifolds in the context of LCK geometry.

Our main result is the following:

Theorem 3.1. *Let X be a compact, complex manifold of complex dimension $n \geq 2$, and let θ be a closed one-form on X. Then X admits a LCK metric with Lee form θ if and only if there are no non-trivial positive currents which are $(1,1)$ components of d_θ-boundaries.*

Remark 3.2. Suppose X is a compact complex manifold, admitting a LCK metric, ω, with Lee form θ. Then any closed 1-form $\eta \in [\theta]_{dR}$ will be a Lee form for a conformal metric of ω and moreover, any conformal change of ω will be LCK with a Lee form in the same de Rham cohomology class as θ. Therefore, we need not fix θ, we can directly use its cohomology class, $[\theta]_{dR}$. By this observation, the theorem above can be rephrased as follows: Let X be a compact, complex manifold of complex dimension $n \geq 2$, and let $[\theta]_{dR}$ a cohomology class in $H^1_{dR}(X)$. Then X admits a LCK metric with Lee form θ if and only if there are no non-trivial positive currents which are $(1,1)$-components of d_η-boundaries, for any closed one-form η belonging to $[\theta]_{dR}$.

4. Details on the paper "Pair correlation of angles between reciprocal geodesics on the modular surface"

The existence of the limiting pair correlation for angles between reciprocal geodesics on the modular surface is established. An explicit formula is provided, which captures geometric information about the length of reciprocal geodesics, as well as arithmetic information about the associated reciprocal classes of binary quadratic forms. One striking feature is the absence of a gap beyond zero in the limiting distribution, contrasting with the analog Euclidean situation.

5. Details on the paper "The Cauchy problems for Einstein metrics and parallel spinors"

This paper aims to solve the problem of extending a spinor from a hypersurface to a parallel spinor on the total space. This problem is related to that of extending a Riemannian metric on a hypersurface to an Einstein metric on the total space, since parallel spinors can only exist over Ricci-flat manifolds.

The Cauchy problem for Einstein metrics. In the Lorentzian setting, Ricci-flat or more generally Einstein metrics form the central objects of general relativity. Given a space-like hypersurface, a Riemannian metric, and a symmetric tensor which plays the role of the second fundamental form, there always exists a local extension to a Lorentzian Einstein metric [17], [15], provided that the local conditions given by the contracted Gauss and Codazzi-Mainardi equations are satisfied. One crucial step in the proof is the reduction to an evolution equation which is (weakly) hyperbolic due to the signature of the metric. The corresponding equations in the Riemannian setting are (weakly) elliptic and no general local existence results are available.

In fact, if (M, g) is any hypersurface of an Einstein manifold $(\mathcal{Z}, g^{\mathcal{Z}})$, then the Weingarten tensor W is a symmetric endomorphism field on M which satisfies certain constraints, which are contractions of the Gauss and Codazzi-Mainardi equations. Conversely, one can ask the following question:

> *(Q1): If W is a symmetric endomorphism field on M which satisfies the constraints, does there exist a isometric embedding of M into a (Riemannian) Einstein manifold $(\mathcal{Z}^{n+1}, g^{\mathcal{Z}})$ with Weingarten tensor W? Is g^Z unique near M up to isometry?*

The uniqueness part is known to have a positive answer by recent results of Biquard [6, Thm. 4] and Anderson-Herzlich [3]. The existence was settled in a paper by Koiso [30] in the real analytic setting. As we were unaware of that paper, in a previous draft of this work we had proved in detail that the answer to the existence part of the above Cauchy problem is positive in the analytic setting. We review the proof and show that the answer is negative, in general, in the smooth setting.

Let us also mention that DeTurck [16] analyzed in the Riemannian setting the somewhat related problem of finding a metric with prescribed nonsingular Ricci tensor.

Extension of generalized Killing spinors to parallel spinors. Our main focus in this paper is the extension problem for spinors. In order to introduce it, we must recall some basic facts about restrictions of spin bundles to hypersurfaces. If $\mathcal{Z}$ is a Riemannian spin manifold, any oriented hypersurface $M \subset \mathcal{Z}$ inherits a spin structure and it is well-known that the restriction to M of the complex spin bundle $\Sigma\mathcal{Z}$ if n is even (resp. $\Sigma^+\mathcal{Z}$ if n is odd) is canonically isomorphic to the complex spin bundle ΣM (cf. [5]). If W denotes the Weingarten tensor of M, the spin covariant derivatives $\nabla^{\mathcal{Z}}$ on $\Sigma\mathcal{Z}$ and ∇^g on ΣM are related by ([5, Eq. (8.1)])

$$ccc \tag{6}$$

for all spinors (resp. half-spinors for n odd) Ψ on $\mathcal{Z}$. We thus see that if Ψ is a parallel spinor on $\mathcal{Z}$, its restriction ψ to any hypersurface M is a *generalized Killing spinor* on M, i.e. it satisfies the equation

$$\nabla^g_X \psi = \tfrac{1}{2} W(X)\cdot\psi, \qquad \forall\ X \in TM, \tag{7}$$

and the symmetric tensor W, called the stress-energy tensor of ψ, is just the Weingarten tensor of the hypersurface M. It is natural to ask whether the converse holds:

> *(Q2): If ψ is a generalized Killing spinor on M^n, does there exist an isometric embedding of M into a spin manifold $(\mathcal{Z}^{n+1}, g^{\mathcal{Z}})$ carrying a parallel spinor Ψ whose restriction to M is ψ?*

This question is the Cauchy problem for metrics with parallel spinors asked in [5].

The answer is known to be positive in several special cases: if the stress-energy tensor W of ψ is the identity [4], if W is parallel [35] and if W is a Codazzi tensor [5]. Even earlier, Friedrich [20] had worked out the 2-dimensional case $n + 1 = 2 + 1$, which is also covered by [5, Thm. 8.1] since on surfaces the stress-energy of a generalized Killing spinor is automatically a Codazzi tensor. Some related embedding results were also obtained by Kim [28], Lawn–Roth [33] and Morel [36]. The common feature of each of these cases is that one can actually construct in an explicit way the "ambient" metric $g^{\mathcal{Z}}$ on the product $(-\varepsilon, \varepsilon) \times M$.

Our aim is to show that the same is true more generally, under the sole additional assumption that (M, g) and W are analytic.

Theorem 5.1. *Let ψ be a spinor field on an analytic spin manifold (M^n, g), and W an analytic field of symmetric endomorphisms of TM. Assume that ψ is a generalized Killing spinor with respect to W, i.e. it satisfies* (7). *Then there exists a unique metric $g^{\mathcal{Z}}$ of the form $g^{\mathcal{Z}} = dt^2 + g_t$, with $g_0 = g$, on a sufficiently small neighborhood $\mathcal{Z}$ of $\{0\} \times M$ inside $\mathbb{R} \times M$ such that $(\mathcal{Z}, g^{\mathcal{Z}})$, endowed with the spin structure induced from M, carries a parallel spinor Ψ whose restriction to M is ψ.*

In particular, the solution $g^{\mathcal{Z}}$ must be Ricci-flat. Einstein manifolds are analytic but of course hypersurfaces can lose this structure so our hypothesis is restrictive. Note that Einstein metrics with smooth initial data can be constructed for small time as constant sectional curvature metrics when the second fundamental form is a Codazzi tensor, see [5, Thm. 8.1]. In particular in dimensions $1+1$ and $2+1$ Theorem 5.1 remains valid in the smooth category since the tensor W associated to a generalized Killing spinor is automatically a Codazzi tensor in dimensions 1 and 2.

The situation changes drastically in higher dimensions for smooth (instead of analytic) generalized Killing spinors. What we can still achieve then is to solve the Einstein equation (and the parallel spinor equation) in Taylor series near the initial hypersurface. More precisely, starting from a smooth hypersurface (M, g) with prescribed Weingarten tensor W we prove that there exist formal Einstein metrics $g^{\mathcal{Z}}$ such that W is the second fundamental form at $t = 0$, i.e., we solve the Einstein equation modulo rapidly vanishing errors. Guided by the analytic and the low dimensional ($n = 1$ or $n = 2$) cases, one could be tempted to guess that actual germs of Einstein metrics do exist for any smooth initial data. However this turns out to be false. Counterexamples were found very recently in some particular cases in dimensions 3 and 7 by Bryant [9]. We give a general procedure to construct counterexamples in all dimensions .

Note that several particular instances of Theorem 5.1 have been proved in recent years, based on the characterization of generalized Killing spinors in terms of exterior forms in low dimensions. Indeed, in dimensions 5, 6 and 7, generalized Killing spinors are equivalent to so-called *hypo*, *half-flat* and *co-calibrated* G_2 structures respectively. In [26] Hitchin proved that the cases $6+1$ and $7+1$ can be solved up to the local existence of a certain gradient flow. Later on, Conti and Salamon [13], [14] treated the cases $5+1$, $6+1$ and $7+1$ in the analytical setting, cf. also [12] for further developments.

A construction related to the Cauchy problem for Einstein metrics has been studied starting with the work of Fefferman-Graham [18] concerning asymptotically hyperbolic Poincaré-Einstein metrics. The starting hypersurface (M^n, g_0) is then at infinite distance from the manifold $\mathcal{Z} = (0, \varepsilon) \times M$, the metric $g^{\mathcal{Z}}$ being conformal to a metric $\bar{g}$ of class C^{n-1} on the manifold with boundary $\overline{\mathcal{Z}} = [0, \varepsilon) \times M$:

$$g^{\mathcal{Z}} = x^{-2}\bar{g}, \qquad \bar{g} = dx^2 + g_x$$

such that the conformal factor x is precisely the distance function to the boundary $x = 0$ with respect to $\bar{g}$. The metric is required to be Einstein of negative curvature up to an error term which vanishes with all derivatives at infinity. Such a metric always exists; when n is odd, it is smooth down to $x = 0$ and its Taylor series at infinity is determined by the initial metric g_0 and the symmetric transverse traceless tensor g_n appearing as coefficient of x^n in g_x, while in even dimensions some logarithmic terms must be allowed, more precisely g_x is smooth as a function of x and $x^n \log x$.

Let us stress that existence results of Einstein metrics with prescribed first fundamental form and Weingarten tensor clearly cannot hold globally in general.

Counterexamples in the smooth setting. In the second part of the paper we apply the existence results from the analytic setting to prove nonexistence of solutions for certain smooth initial data in any dimension at least 3.

The argument goes along the lines of works of the first author and his collaborators on the Yamabe problem and the mass endomorphism. We consider the functional

$$\mathcal{F}(\phi) := \frac{\langle D_0\phi, \phi\rangle_{L^2}}{\|D_0\phi\|^2_{L^{2n/(n+1)}}}$$

defined on the C^1 spinor fields ϕ on a compact connected Riemannian spin manifold (M, g_0) which are not in the kernel of the Dirac operator D_0. If the infimum of the lowest positive eigenvalue of the Dirac operator in the volume-normalized conformal class of g_0 is strictly lower than the corresponding eigenvalue for the standard sphere, this functional attains its supremum in a spinor ψ_0 of regularity $C^{2,\alpha}$. Moreover, ψ_0 is smooth outside its zero set.

To construct g_0 we fix $p \in M$ and we look at metrics on M which are flat near p. If the topological index of M vanishes in $KO^{-n}(pt)$, then for generic such metrics the associated Dirac operator is invertible. The mass endomorphism at p is defined as the constant term in the asymptotic expansion of the Green kernel of D near p. Again for generic metrics, this mass endomorphism is non-zero, which by a result of [1] ensures the technical Condition for generic metrics which are flat near p. By construction this class of metrics contains metrics which are not conformally flat on some open subset of M, i.e., whose Schouten tensor (in dimension 3), resp. Weyl curvature (in higher dimensions) is nonzero on some open set. We assume g_0 was chosen with these properties.

We return now to the spinor ψ_0 maximizing the functional $\mathcal{F}$. The Euler-Lagrange equation of $\mathcal{F}$ at ψ_0 can be reinterpreted as follows: the Dirac operator with respect to the conformal metric $g := |\psi_0|^{4/(n-1)} g_0$ admits an eigenspinor of constant length 1, $\psi := \frac{\psi_0}{|\psi_0|}$.

If the dimension n equals 3, by algebraic reasons this spinor field must be a generalized Killing spinor with stress-energy tensor W of constant trace.

The metric g is defined on the complement M^* of the zero set of ψ_0. This set is open, connected and dense in M. Recall that g_0 was chosen such that its Schouten tensor vanishes identically on an open set of M and is nonzero on another open set. Then the same remains true on M^*, and therefore on M^* there exists no analytic metric in the conformal class of g_0. In particular, the metric $g = |\psi_0|^{4/(n-1)} g_0$ cannot be analytic.

Assuming now that the existence theorem continues to hold for smooth initial data, we could apply it to (M^*, g, W) to get an embedding in a Ricci-flat (hence analytic) Riemannian manifold $(\mathcal{Z}, g^{\mathcal{Z}})$, with second fundamental form W. Since the trace of W is constant by construction, M would have constant mean curvature, which would imply that it were analytic, contradicting the non-analyticity proved above.

The above construction actually yields counterexamples to the Cauchy problem for Ricci-flat metrics in the smooth setting in any dimension $n \geq 3$, by taking products with flat spaces.

6. Details on the paper "The renormalized volume and uniformisation of conformal structures"

We study the renormalized volume of asymptotically hyperbolic Einstein (AHE in short) manifolds (M, g) when the conformal boundary ∂M has dimension n even. Its definition depends on the choice of metric h_0 on ∂M in the conformal class at infinity determined by g, we denote it by $\mathrm{Vol}_R(M, g; h_0)$. We show that $\mathrm{Vol}_R(M, g; \cdot)$ is a functional admitting a "Polyakov type" formula in the conformal class $[h_0]$ and we describe the critical points as solutions of some non-linear equation $v_n(h_0) =$ constant, satisfied in particular by Einstein metrics. In dimension $n = 2$, choosing extremizers in the conformal class amounts to uniformizing the surface, while in dimension $n = 4$ this amounts to solving the σ_2-Yamabe problem. Next, we consider the variation of $\mathrm{Vol}_R(M, \cdot; \cdot)$ along a curve of AHE metrics g^t with boundary metric h_0^t and we use this to show that, provided conformal classes can be (locally) parametrized by metrics h solving $v_n(h) = \int_{\partial M} v_n(h) \mathrm{dvol}_h$, the set of ends of AHE manifolds (up to diffeomorphisms isotopic to the identity) can be viewed as a Lagrangian submanifold in the cotangent space to the space $\mathcal{T}(\partial M)$ of conformal structures on ∂M. We obtain as a consequence a higher-dimensional version of McMullen's quasifuchsian reciprocity. We finally show that conformal classes admitting negatively curved Einstein metrics are local minima for the renormalized volume for a warped product type filling.

7. Details on the paper "Positivity of the renormalized volume of almost-Fuchsian hyperbolic 3-manifolds "

The renormalized volume Vol_R is a numerical invariant associated to an infinite-volume Riemannian manifold with some special structure near infinity, extracted from the divergent integral of the volume form. Early instances of renormalized volumes appear in Henningson–Skenderis [25] for asymptotically hyperbolic Einstein metrics, and in Krasnov [31] for Schottky hyperbolic 3-manifolds. In Takhtajan–Teo [41] the renormalized volume is identified to the so-called Liouville action functional, a cohomological quantity known since the pioneering work of Takhtajan–Zograf [43] to be a Kähler potential for the Weil–Petersson symplectic form on the deformation space of certain Kleinian manifolds:

$$\partial\overline{\partial}\mathrm{Vol}_R = \frac{1}{8i}\omega_{\mathrm{WP}}. \tag{8}$$

Krasnov–Schlenker [37] studied the renormalized volume using a geometric description in terms of foliations by equidistant surfaces. In the context of quasi-Fuchsian hyperbolic 3-manifolds they computed the Hessian of Vol_R at the Fuchsian locus. They also gave a direct proof of the identity (8) in that setting. Recently, Guillarmou–Moroianu [23] studied the renormalized volume Vol_R in a general context, for geometrically finite hyperbolic 3-manifolds without rank-1 cusps. There, Vol_R appears as the log-norm of a holomorphic section in the Chern–Simons line bundle over the Teichmüller space.

Huang–Wang [27] looked at renormalized volumes in their study of almost-Fuchsian hyperbolic 3-manifolds. However, their renormalization procedure does not involve uniformization of the surfaces at infinity, hence the invariant RV thus obtained is constant (and negative) on the moduli space of almost-Fuchsian metrics.

There is a superficial analogy between Vol_R and the mass of asymptotically Euclidean manifolds. Like in the positive mass conjecture, one may ask if Vol_R is positive for all convex co-compact hyperbolic 3-manifolds, or at least for quasi-Fuchsian manifolds. One piece of supporting evidence follows from the computation by Takhtajan–Teo [41] of the variation of Vol_R (or equivalently, of the Liouville action functional) on deformation spaces. In the setting of quasi-Fuchsian manifolds, Krasnov–Schlenker [37] noted that the functional Vol_R vanishes at the Fuchsian locus. When one component of the boundary is kept fixed, the only critical point of Vol_R is at the unique Fuchsian metric. Moreover, this point is a local minimum because the Hessian of Vol_R is positive definite there as it coincides with the Weil-Petersson metric. Therefore, at least in a neighborhood of the Fuchsian locus, we do have positivity. We emphasize that to ensure vanishing of the renormalized volume for Fuchsian manifolds, the renormalization procedure used in Krasnov–Schlenker [37] differs from Guillarmou–Moroianu [23] or from Huang–Wang [27] by the universal constant $2\pi(1-g)$ where $g \geq 2$ is the genus. It is the definition from Krasnov–Schlenker [37] that we use below. These results are not sufficient to conclude that Vol_R is positive since the Teichmüller space is not compact and Vol_R is not proper (by combining the results in Schlenker [40] and Brock [8], one sees that the difference between Vol_R and the Teichmüller distance is bounded, while the Teichmüller metric is incomplete). Another piece of evidence towards positivity was recently found by Schlenker [40], who proved that Vol_R is bounded from below by some explicit (negative) constant.

In this paper we prove the positivity of Vol_R on the almost-Fuchsian space, which is an explicit open subset of the space of quasi-Fuchsian metrics.

References

[1] B. Ammann, E. Humbert, B. Morel, *Mass endomorphism and spinorial Yamabe type problems*, Comm. Anal. Geom. **14** (2006), 163–182.

[2] Bernd Ammann, Andrei Moroianu, Sergiu Moroianu The Cauchy problems for Einstein metrics and parallel spinors *Communications in Mathematical Physics* **320**, 173–198 (2013)

[3] M. T. Anderson, M. Herzlich, *Unique continuation results for Ricci curvature and applications*, J. Geom. Phys. **58** (2008), no. 2, 179–207.

[4] C. Bär, *Real Killing spinors and holonomy*, Commun. Math. Phys. **154** (1993), 509–521.

[5] C. Bär, P. Gauduchon, A. Moroianu, *Generalized Cylinders in Semi-Riemannian and Spin Geometry,* Math. Z. **249** (2005), 545–580.

[6] O. Biquard, *Continuation unique à partir de l'infini conforme pour les métriques d'Einstein,* Math. Res. Lett. **15** (2008), no. 6, 1091–1099.

[7] Florin P. Boca, Vicenţiu Paşol, Alexandru A. Popa, Alexandru Zaharescu Pair corellation of angles between reciprocal geodesics on the modular surface *acceptat în Algebra and Number Theory*

[8] J. Brock, *The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores,* J. Amer. Math. Soc. **16** (2003), no. 3, 495–535 (electronic).

[9] R. Bryant, *Non-embedding and non-extension results in special holonomy*, The many facets of geometry – A tribute to Nigel Hitchin, 346–367, Oxford Univ. Press, Oxford, 2010.

[10] U. Bunke, *String structures and trivialisations of a Pfaffian line bundle*, Comm. Math. Phys. **307** (2011), no. 3, 675–712.

[11] S.-S. Chern, J. Simons, *Characteristic forms and geometric invariants*, Ann. Math. **99** (1974), 48–69.

[12] D. Conti, *Embedding into manifolds with torsion*, Math. Z. **268** (2011), no. 3-4, 725–751.

[13] D. Conti, S. Salamon, *Reduced holonomy, hypersurfaces and extensions*, Int. J. Geom. Methods Mod. Phys. **3** (2006), no. 5-6, 899–912.

[14] ______, *Generalized Killing spinors in dimension 5*, Trans. Amer. Math. Soc. **359** (2007), no. 11, 5319–5343.

[15] D. DeTurck, *The Cauchy problem for Lorentz metrics with prescribed Ricci curvature*, Compositio Math. **48** (1983), no. 3, 327–349.

[16] ______, *Existence of metrics with prescribed Ricci curvature: local theory,* Invent. Math. **65** (1981/82), no. 1, 179–207.

[17] Y. Fourès-Bruhat, *Résolution du problème de Cauchy pour des équations hyperboliques du second ordre non linéaires,* Bull. Soc. Math. France **81** (1953), 225–288.

[18] C. L. Fefferman, C. R. Graham, *Conformal invariants*, Asterisque, vol. hors série, Soc. Math. France, (1985), 95–116.

[19] D. S. Freed, *Classical Chern-Simons theory, Part 1*, Adv. Math. **113** (1995), 237–303.

[20] Th. Friedrich, *On the spinor representation of surfaces in Euclidean* 3*-space*, J. Geom. Phys. **28** (1998), 143–157.

[21] C. R Graham, *Volume and area renormalizations for conformally compact Einstein metrics*, Rend. Circ. Mat. Palermo Ser. II **63** (2000), Suppl., 31–42.

[22] Colin Guillarmou, Sergiu Moroianu, Jinsung Park Bergman and Calderón projectors for Dirac operators *Journal of Geometric Analysis* **24**(1), 298–336 (2014)

[23] Colin Guillarmou, Sergiu Moroianu, Chern-Simons line bundle on Teichmüller space *acceptat în Geometry and Topology*

[24] Corina Ciobotaru, Sergiu Moroianu Positivity of the renormalized volume of almost-Fuchsian hyperbolic 3-manifolds *preprint arXiv:1402.2555*

[HL] R. Harvey, H.B. Lawson Jr., *An intrinsic characterization of Kähler manifolds*, Invent. Math. **74** (1983), 169–198.

[25] M. Henningson, K. Skenderis, *The holographic Weyl anomaly*, J. High Energy Phys. (1998), no. 7, paper 23, 12 pp. (electronic).

[26] N. Hitchin, *Stable forms and special metrics*, Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000), 70–89, Contemp. Math. **288**, Amer. Math. Soc. Providence, RI, 2001.

[27] Zheng Huang, Biao Wang, *On almost-fuchsian manifolds,* Trans. Amer. Math. Soc. **365** (2013), no. 9, 4679–4698.

[28] E. C. Kim, *A local existence theorem for the Einstein-Dirac equation*, J. Geom. Phys. **44**, (2002), 376–405.

[29] P. Kirk, E. Klassen, *Chern-Simons invariants of* 3*-manifolds decomposed along tori and the circle bundle over the representation space of* T^2, Comm. Math. Phys. **153** (1993), no. 3, 521–557.

[30] N. Koiso, *Hypersurfaces of Einstein manifolds.* Ann. Sci. École Norm. Sup. **14** (1981), no. 4, 433–443.

[31] K. Krasnov, *Holography and Riemann surfaces*, Adv. Theor. Math. Phys. **4** (2000) no. 4, 929–979.

[32] K. Krasnov, *On Holomorphic Factorization in Asymptotically AdS 3D Gravity*, Class. Quant. Grav. **20** (2003), 4015–4042.

[33] M.-A. Lawn, J. Roth, *Isometric immersions of hypersurfaces in 4-dimensional manifolds via spinors*, Diff. Geom. Appl. **28** (2010), no. 2, 205–219.

[34] A. Marden, *Deformation of Kleinian groups*, Chap. 9, Handbook of Teichmüller theory, Vol. I, edited by A. Papadopoulos, IRMA Lect. Math. Theor. Phys. 11, Eur. Math. Soc. (2007).
[35] B. Morel, *The energy-momentum tensor as a second fundamental form*, math.DG/0302205.
[36] B. Morel, *Surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$ via spinors*, Actes de Séminaire de Théorie Spectrale et Géométrie, **23**, Anné 2004–2005, 131–144, Univ. Grenoble I, Saint-Martin-d'Hères, 2005.
[37] K. Krasnov, J.-M. Schlenker, *On the Renormalized Volume of Hyperbolic 3-Manifolds*, Commun Math. Phys. **279** (2008), no. 3, 637–668.
[38] D. Quillen, *Determinants of Cauchy-Riemann operators over a Riemann surface*. Funct. Anal. Appl. **19** (1986), 31–34.
[39] T. R. Ramadas, I. M. Singer, J. Weitsman, *Some comments on Chern-Simons gauge theory*, Commun. Math. Phys. **126**, 409–420 (1989).
[40] J.-M. Schlenker, *The renormalized volume and the volume of the convex core of quasifuchsian manifolds*, Math. Res. Lett. **20** (2013), 773–786.
[41] L. A. Takhtajan, L-P. Teo, *Liouville action and Weil-Petersson metric on deformation spaces, global Kleinian reciprocity and holography*, Comm. Math. Phys. **239** (2003) no. 1-2 , 183–240.
[42] L. A. Takhtajan, L.-P. Teo, *Liouville action and Weil-Petersson metric on deformation spaces, global Kleinian reciprocity and holography.* Comm. Math. Phys. **239** (2003), no. 1–2, 183–240.
[43] L. A. Takhtadzhyan, P. G. Zograf, *On the uniformization of Riemann surfaces and on the Weil-Petersson metric on the Teichmüller and Schottky spaces*, Mat. Sb. (N.S.) **132** (174) (1987), no. 3, 304–321, 444; translation in Math. USSR-Sb. 60 (1988), no. 2, 297–313.
[44] S. Wolpert, *Chern forms and the Riemann tensor for the moduli space of curves*, Invent. Math. **85** (1986), 119–145.
[45] T. Yoshida, *The η-invariant of hyperbolic* 3*-manifolds*, Invent. Math. **81** (1985), 473–514.
[46] P. G. Zograf, *Liouville action on moduli spaces and uniformization of degenerate Riemann surfaces*, Algebra i Analiz **1** (1989), no. 4, 136–160 (Russian), English translation in Leningrad Math. J. **1** (1990), no. 4, 941–965.
[47] P. G. Zograf, *Determinants of Laplacians, Liouville action, and an analogue of the Dedekind η-function on Teichmüller space*, Unpublished manuscript (1997).

Sergiu Moroianu, Institutul de Matematică al Academiei Române, P.O. Box 1-764, RO-014700 Bucharest, Romania
E-mail address: moroianu@alum.mit.edu