FMI, Info, Master I
Semestrul I 2014/2015
Tehnici de optimizare combinatorială
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Tema 5

În continuare, dimensiunea unei mulţimi finite este cardinalul ei. De asemenea,

digraf	=	digraph
tăietură	=	cut
drum	=	path
cuplaj	=	matching
acoperire cu vârfuri	=	vertex cover
mulţime de arce care deconectează s, t	=	s-t disconnecting arc set

(H5.1) Fie $D = (V, A)$ un digraf şi $s, t \in V$. Definim

$$\mathcal{D} = \{B \subseteq A \mid B \text{ este mulţime de arce care deconectează } s, t\}.$$

Demonstraţi că:

(i) Orice mulţime din $\mathcal{D}$ de dimensiune minimă este s-t tăietură.

(ii) Dimensiunea minimă a unei mulţimi din $\mathcal{D}$ coincide cu dimensiunea minimă a unei s-t tăieturi.

(Aceastea sunt punctele (ii),(iii) din Lema 4.1.5 din notele de curs.)

Proof. (i) Let U be the set of vertices in V accessible from s by paths that contain no arcs of B. Then $s \in U$ and $t \notin U$, since B is s-t disconnecting. Hence, $\delta^{out}(U)$ is an s-t cut, so it is s-t disconnecting, by (S6.5).

Claim: $\delta^{out}(U) \subseteq B$

Proof of Claim: Let $a = (u, v) \in \delta^{out}(U)$. Since $u \in U$, there exists an s-u path P containing no arcs of B. If $a \notin B$, then $P + a$ is an s-t path P containing no arcs of B, hence $t \in U$, which is a contradiction. Thus, $a \in B$. ∎

By the fact that B is of minimum size, we must have $B = \delta^{out}(U)$.

(ii) Let m_1 be the first minimum and m_2 be the second minimum. We have that $m_1 \leq m_2$ by (S6.5) and that $m_1 \geq m_2$ by (i).

□

(H5.2) Fie $D = (V, A)$ un digraf şi $s, t \in V$ cu proprietatea că există s-t drumuri. *Lungimea* unui s-t drum P, notată cu $\ell(P)$, este numărul arcelor sale. *Distanţa* de la s la t, notată cu $d(s, t)$, este lungimea minimă a unui s-t drum. Fie N numărul s-t tăieturilor disjuncte din D. Demonstraţi că:

(i) $\ell(P) \geq N$ pentru orice s-t drum P.

(ii) $d(s, t) = N$.

Proof. (i) For any s-t cut C_i, $i = 1, \ldots, N$ there exists an arc $a_i \in P \cap C_i$, as C_i is an s-t disconnecting arc set, by (S6.5). Since the s-t cuts C_i, $i = 1, \ldots, N$ are disjoint, we have that the arcs $a_i, i = 1, \ldots, N$ are different. It follows that $\ell(P) \geq N$.

(ii) Let us denote, for simplicity, $d := d(s, t)$. We have to prove that $d = N$.

"$\geq$" By (i).

"$\leq$" For every $i = 1, \ldots, d$, let $U_i := \{v \in A \mid d(s, v) < i\}$. Then $s \in U_i$ and $t \notin U_i$, hence $C_i := \delta^{out}(U_i)$ is an s-t cut.

Claim: For every arc $(u, v) \in C_i$, we have that $d(s, u) = i - 1$ and $d(s, v) = i$.

Proof of Claim: If $a = (u, v) \in C_i$, then $a \in U_i$ and $v \notin U_i$, hence $d(s, u) < i$ and $d(s, v) \geq i$. If P is an s-u path of length $d(s, u)$, then $P \subseteq U_i$, hence $v \notin P$. It follows that $P + (u, v)$ is an s-v path. As a consequence, $d(s, v) \leq d(s, u) + 1$. We have got that $i \leq d(s, v) \leq d(s, u) + 1 \leq i$. Thus, both inequalities must be equalities. ■

As an immediate consequence, we get that for $i < j$, if $a = (u, v) \in C_i$, then $d(s, v) = i < j$, hence $v \in U_j$, so $a \notin C_j$. We have proved that C_i and C_j are disjoint. It follows that there are at least d s-t cuts, hence $N \geq d$.

□

(H5.3) Fie G un graf, $\nu(G)$ dimensiunea maximă a unui cuplaj în G şi $\tau(G)$ dimensiunea minimă a unei acoperiri cu vârfuri în G. Demonstraţi că $\nu(G) \leq \tau(G)$. Daţi exemplu de un graf în care are loc inegalitatea strictă (adică $\nu(G) < \tau(G)$).

Proof. Let $M = \{e_1, \ldots, e_n\}$ and S be a vertex cover of G. Then for every $i = 1, \ldots,$ e_i intersects S in some $v_i \in V$. Since the edges in M are disjoint, it follows that for different i's we get different v_i's. Thus $S \geq n$. It follows that $\nu(G) \leq \tau(G)$.
An example where strict inequality holds is the graph $K_3 = (\{1, 2, 3\}, \{12, 23, 31\}$. Then $\nu(K_3) = 1 < 2 = \tau(K_3)$. □

(H5.4) Se consideră următoarea matrice

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Să se verifice dacă A este total unimodulară.

Proof. A is the incidence matrix of the bipartite graph $G = (X \cup Y, E)$, where

$$X = \{1, 2, 3\}, \quad Y = \{4, 5, 6\}, E = \{14, 15, 25, 26, 36, 3, 4\}.$$

Thus, A is totally unimodular, by Theorem 2.3.8. □

Spunem că o s-t tăietură B a unei reţele de flux $N = (D, c, s, t)$ este *minimală* dacă B are capacitate minimă.

(H5.5) Fie $N = (D, c, s, t)$ of reţea de flux, B o s-t tăietură minimală şi $\lambda > 0$. Fie N_1 reţeaua obţinută din N multiplicând toate capacităţile arcelor cu λ şi N_2 reţeaua obţinută din N adunând λ la toate capacităţile arcelor.

(i) Este B s-t tăietură minimală în N_1?

(ii) Este B s-t tăietură minimală în N_2?

Justificaţi în fiecare caz răspunsul printr-o demonstraţie sau un contraexemplu.

Proof. (i) The answer is YES. We have that for any s-t cut $C \subseteq D$, $c_{N_1}(C) = \lambda c_N(C)$. Thus, the relative order of s-t cuts does not change.

(ii) The answer is NO. Let us consider the network N given in Figure 1.

Then the minimal cut is $B = \delta^{out}(\{1, 2, 3\})$ with $c_N(B) = 2$.

Take $\lambda = 2$. Then N_2 is given in given in Figure 2.

Then $c_{N_2}(B) = 8$, which is not anymore minimum, since if we take the cut $B' := \delta^{out}(\{s\})$, we have that $c_{N_2}(B') = 6$. □

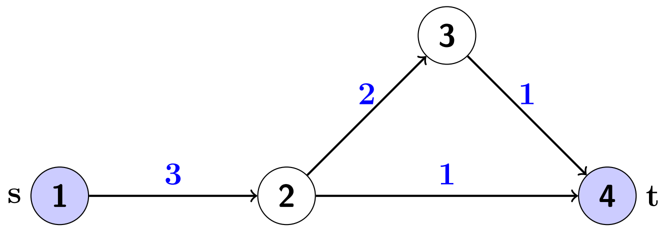

Figure 1: Reţeaua de flux N

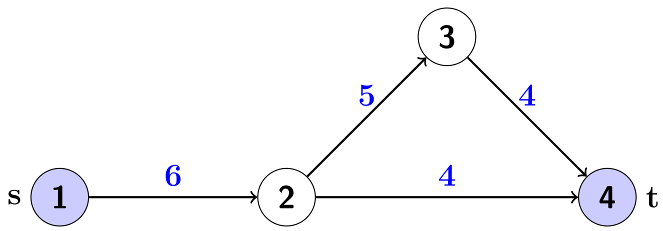

Figure 2: Reţeaua de flux N_2