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Summary of PhD Thesis

Optimal Control Problems for Stochastic Differential Equations

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CONTENTS

Key words: Stochastic Optimal Control Problem; Stochastic Differential Equations; Feedback Control; Deterministic Optimal Control Problem; Open-loop Control; Kolmogorov Equations; Fokker-Planck Equations; Weak Solutions to PDEs; Optimality Conditions; Semigroups of Linear Operators.

1. INTRODUCTION

This work concerns some stochastic optimal control problems with feedback inputs. We deal in the present thesis with a new method in the mathematical literature which consists in considering an equivalent or related deterministic optimal control problem for certain (forward or backward) Kolmogorov equations. The information we get about the deterministic problems gives deeper insight into the stochastic ones.

In this chapter we present the background of the problems investigated in this thesis and outline the original contribution of the author. Here is the structure of this thesis:

Chapter 2: Stochastic optimal control problems with feedback inputs via Kolmogorov equations. This chapter concerns a certain type of stochastic optimal control problems with feedback inputs and is based on the author's papers [1] and [3]. The results were presented in two talks (see [C1], [C2]) given at International Conferences.

Firstly we show that there is a deep relationship between the stochastic problem and a deterministic optimal control problem for a couple of (backward) Kolmogorov equations with open-loop controllers. One proves the existence of an optimal control for the deterministic problem if the drift coefficient has a particular form and if a certain convexity property for the cost functional holds. A maximum principle is proved and some first order necessary optimality conditions are obtained. We give some examples and discuss an alternative semigroup approach. We emphasize that the method can be adapted to investigate an even more general class of stochastic optimal control problems. A few auxiliary results are presented at the end of the chapter.

Chapter 3: Optimal control of stochastic differential equations via Fokker-Planck equations. This chapter concerns an optimal control problem with feedback (closed-loop) inputs related to a stochastic differential equation and an associated deterministic optimal control problem with open-loop controllers for a certain Fokker-Planck equation (forward Kolmogorov equation). This chapter is based on the author's papers [2] and [4]. Some of the ideas and results were presented in [C3] and [C4].

Some basic properties of the weak solutions to a stochastic differential equation and to a Fokker-Planck equation are investigated and the relationship between the two equations is discussed. Under certain assumptions, the equivalence between the stochastic and the deterministic optimal control problems is proven. The superposition principle is one of the main tools in the proof of the equivalence. A maximum principle is established using the so-called spike controls and necessary optimality conditions for the deterministic problem are derived. One obtains a similar result in the case of time-independent controllers. The existence of an optimal control is proven under additional hypotheses for the deterministic optimal control problem. Some examples illustrate the applicability of the theoretical results. One discusses some aspects concerning a problem with a control with nonlocal action. This chapter ends with some auxiliary results.

Chapter 4: Further extensions. This short chapter suggests some possible extensions and topics for future investigation. A special attention is paid to the control of the Fokker-Planck equation with nonlocal term. Another extension concerns an optimal control problem of a nonlinear Fokker-Planck equation and its relationship with a stochastic optimal control problem of the McKean-Vlasov equation.

Some of these subjects are currently under investigation in [4].

Chapter 5: Appendix. Here we recall a few notions and results that are indispensable throughout this PhD thesis: Gronwall's inequality, Lions' existence theorem, Aubin's compactness theorem, and the theorems of Lumer-Phillips and of Trotter-Kato concerning the C_0-semigroups.

The original results of the author are contained in chapters 2 and 3, while further possible extensions are included in chapter 4.

Some of the original results have been communicated to international conferences:

C1. Ş.-L. Aniţa, Optimal control for SDEs with feedback inputs and related Kolmogorov equations, Atelier de travail en Stochastique et EDP, Bucharest, Romania, 20 October, 2020, `http://imar.ro/CFM/2020/EDP-Stochastique-Oct2020.pdf`;

C2. Ş.-L. Aniţa, Stochastic optimal control problems and related Kolmogorov equations, IWSPA 2020 - International Workshop on Stochastic Processes and Their Applications, A virtual workshop, 24 November - 9 December, 2020, `http://blogs.mat.ucm.es/presa/program2020/`;

C3. Ş.-L. Aniţa, Optimal control problem for McKean-Vlasov stochastic equation, The 10th International Conference on Stochastic Analysis and its Applications (10th ICSAA), Kyoto University, Japan, 6-10 September, 2021, `https://www.math.kyoto-u.ac.jp/workshop/ICSAA2020/ICSAA2020_program.pdf` ;

C4. Ş.-L. Aniţa, An optimal control problem related to a non-linear Fokker-Planck equation, Young Researchers Workshop - Romanian Society of Probability and Statistics, Bucharest, Romania, 19 November, 2021, `https://spsr.ase.ro/wp-content/uploads/2021/10/spsr-workshop-2021-3.pdf`.

2. STOCHASTIC OPTIMAL CONTROL PROBLEMS WITH FEEDBACK INPUTS VIA KOLMOGOROV EQUATIONS

2.1 Formulation of the problem

Consider the following stochastic optimal control problem with feedback inputs

$$(\mathbf{CP_S}) \qquad \underset{u\in\mathcal{M}_c}{\text{Minimize}} \left\{ \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} G(X^u(t,x),u(X^u(t,x)))d\nu(x)\ dt\right] + \mathbb{E}\left[\int_{\mathbb{R}^d} G_T(X^u(T,x))d\nu(x)\right]\right\},$$

where X^u is the solution to

$$\begin{cases} dX(t) = f(X(t),u(X(t)))dt + \sigma(X(t))dW(t), \quad t\in[0,T] \\ \\ X(0) = x\in\mathbb{R}^d. \end{cases} \tag{2.1}$$

Here $T\in(0,+\infty)$, $d,\ n,\ m\in\mathbb{N}^*$, $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space, $W:[0,T]\times\Omega\to\mathbb{R}^n$ is a Wiener process, and $(\mathcal{F}_t)_{t\in[0,T]}$ is the corresponding natural filtration.

$$\mathcal{M}_c = \{v\in C_b^2(\mathbb{R}^d;\mathbb{R}^m); v(x)\in U_0,\ \forall x\in\mathbb{R}^d\}$$

is the set of controllers, and U_0 is a bounded convex and closed subset of $\mathbb{R}^m$ with $0_m\in U_0$.

(H2.1) ν is a finite measure on $\mathbb{R}^d$ with a density ρ which satisfies

$$\rho \in C_b^1(\mathbb{R}^d), \qquad \rho(x) > 0, \ \forall x \in \mathbb{R}^d, \qquad \frac{\nabla\rho}{\rho} \in C_b(\mathbb{R}^d; \mathbb{R}^d).$$

The functions $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\cdot n}$, $G : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$, $G_T : \mathbb{R}^d \to \mathbb{R}$, $f(x,u) = (f_1(x,u)\ f_2(x,u)\ ...\ f_d(x,u))^T$, $\sigma(x) = (\sigma_{il}(x))_{i=1,2,...,d,\ l=1,2,...,n}$, $q(x) = (q_{ij}(x))_{i=1,2,...,d,\ j=1,2,...,d} = \sigma(x)\sigma(x)^T$, $\forall x \in \mathbb{R}^d, u \in \mathbb{R}^m$, satisfy

(H2.2) $f|_{\mathbb{R}^d\times\tilde{U}_0}$ is bounded and Lipschitz continuous on $\mathbb{R}^d \times \tilde{U}_0$, where $\tilde{U}_0$ is an open neighborhood of U_0;

(H2.3) $\sigma \in C_b^1(\mathbb{R}^d; \mathbb{R}^{d\cdot n})$ and there exists a constant $\gamma > 0$ such that

$$q_{ij}(x)y_iy_j = \sigma(x)\sigma(x)^T y \cdot y \geq \gamma|y|_d^2, \quad \forall x, y \in \mathbb{R}^d;$$

(H2.4) $G|_{\mathbb{R}^d\times\tilde{U}_0} \in C_b(\mathbb{R}^d \times \tilde{U}_0)$ and $G_T \in C_b(\mathbb{R}^d)$.

For any $u \in \mathcal{M}_c$ we have that $G(\cdot, u(\cdot)), G_T \in C_b(\mathbb{R}^d)$ and that the functions $\varphi_1^u, \varphi_2^u : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}$,

$$\varphi_1^u(t,x) = \mathbb{E}[G(X^u(t,x), u(X^u(t,x)))], \quad \varphi_2^u(t,x) = \mathbb{E}[G_T(X^u(t,x))], \quad (t,x) \in [0,T] \times \mathbb{R}^d, \tag{2.2}$$

are the unique weak solutions (for definition see [1]) to

$$\begin{cases} \dfrac{\partial\varphi_1}{\partial t}(t,x) = f(x,u(x)) \cdot \nabla\varphi_1(t,x) + \dfrac{1}{2}q_{ij}(x)\dfrac{\partial^2\varphi_1}{\partial x_i\partial x_j}(t,x), & x \in \mathbb{R}^d,\ t \in (0,T) \\ \varphi_1(0,x) = G(x,u(x)) = \varphi_{01}^u(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.3}$$

and

$$\begin{cases} \dfrac{\partial\varphi_2}{\partial t}(t,x) = f(x,u(x)) \cdot \nabla\varphi_2(t,x) + \dfrac{1}{2}q_{ij}(x)\dfrac{\partial^2\varphi_2}{\partial x_i\partial x_j}(t,x), & x \in \mathbb{R}^d,\ t \in (0,T) \\ \varphi_2(0,x) = G_T(x) = \varphi_{02}(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.4}$$

respectively. It is obvious that for any $u \in \mathcal{M}_c$ we have that

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} G(X^u(t,x), u(X^u(t,x)))d\nu(x)\ dt\right] + \mathbb{E}\left[\int_{\mathbb{R}^d} G_T(X^u(T,x))d\nu(x)\right]$$

$$= \int_0^T \int_{\mathbb{R}^d} \varphi_1^u(t,x)d\nu(x)\ dt + \int_{\mathbb{R}^d} \varphi_2^u(T,x)d\nu(x),$$

and that (CP_S) is equivalent to the following deterministic optimal control problem with open-loop controllers

$$(\mathbf{CP_D}) \qquad \underset{u\in\mathcal{M}_c}{\text{Minimize}} \left\{\int_0^T \int_{\mathbb{R}^d} \varphi_1^u(t,x)d\nu(x)\ dt + \int_{\mathbb{R}^d} \varphi_2^u(T,x)d\nu(x)\right\}.$$

It is convenient to consider a larger set of controllers

$$\mathcal{M} = \{v \in L^\infty(\mathbb{R}^d; \mathbb{R}^m); v(x) \in U_0 \text{ a.e. } x \in \mathbb{R}^d\},$$

and the deterministic optimal control problem

$$(\mathbf{CP}) \qquad \underset{u\in\mathcal{M}}{\text{Minimize}} \left\{\int_0^T \int_{\mathbb{R}^d} \varphi_1^u(t,x)d\nu(x)\ dt + \int_{\mathbb{R}^d} \varphi_2^u(T,x)d\nu(x)\right\},$$

where φ_1^u and φ_2^u are the weak solutions to (2.3) and (2.4), respectively.

2.2 Relationship between the stochastic and the deterministic optimal control problems

Consider the following real vector spaces: $H = L^2(\mathbb{R}^d; \nu) = \{\psi \in L^2_{loc}(\mathbb{R}^d);\ \sqrt{\rho}\psi \in L^2(\mathbb{R}^d)\}$,

$$V = W^{1,2}(\mathbb{R}^d; \nu) = \left\{\psi \in W^{1,2}_{loc}(\mathbb{R}^d);\ \psi, \frac{\partial \psi}{\partial x_i} \in L^2(\mathbb{R}^d; \nu),\ i = 1, 2, ..., d\right\}.$$

By $(H2.1)$ it follows that $V = \{\psi \in W^{1,2}_{loc}(\mathbb{R}^d); \sqrt{\rho}\psi \in W^{1,2}(\mathbb{R}^d)\}$.

Notice that $(H, \langle\cdot,\cdot\rangle_H)$ and $(V, \langle\cdot,\cdot\rangle_V)$ are real Hilbert spaces, where

$$\langle\varphi, \psi\rangle_H = \int_{\mathbb{R}^d} \varphi\psi\ d\nu(x) = \int_{\mathbb{R}^d} \varphi\psi\rho\ dx,$$

$$\langle\varphi, \psi\rangle_V = \int_{\mathbb{R}^d} [\varphi\psi + \nabla\varphi \cdot \nabla\psi]\ d\nu(x) = \int_{\mathbb{R}^d} [\varphi\psi + \nabla\varphi \cdot \nabla\psi]\rho\ dx$$

are their scalar products. We identify the dual of H (i.e. H^*) with H and denote by V^* the dual of V with the pairing denoted by $\langle\cdot,\cdot\rangle_{V,V^*}$ or $\langle\cdot,\cdot\rangle_{V^*,V}$. Moreover, $\langle\varphi,\psi\rangle_{V,V^*} = \langle\varphi,\psi\rangle_H$, for any $\varphi \in V, \psi \in H$. This yields $V \subset H \subset V^*$ with continuous and dense embeddings.

Note that there exist two positive constants m_0, M_0 such that

$$m_0\|\varphi\|_V \leq \|\varphi\sqrt{\rho}\|_{W^{1,2}(\mathbb{R}^d)} \leq M_0\|\varphi\|_V, \quad \forall\varphi \in V.$$

Let us now investigate the following Cauchy problem

$$\begin{cases} \dfrac{d\phi}{dt}(t) = \mathcal{A}_0\phi(t) + g(t), \quad t \in (0, T) \\ \\ \phi(0) = \phi_0, \end{cases} \tag{2.5}$$

where $\mathcal{A}_0 \in L(V, V^*)$ is given by $\langle\mathcal{A}_0 v_1, v_2\rangle_{V^*,V} = -a^0(v_1, v_2)$, $\forall v_1, v_2 \in V$ and $a^0 : V \times V \longrightarrow \mathbb{R}$ is bilinear and bounded and $g \in L^2(0, T; V^*)$.

By Lions' existence theorem we get that if $\phi_0 \in H$ and a^0 satisfies in addition that

$$\exists\alpha^0 > 0, \beta^0 \geq 0 : \quad a^0(v, v) \geq \alpha^0\|v\|^2_V - \beta^0\|v\|^2_H, \quad \forall v \in V, \tag{2.6}$$

then (2.5) has a unique weak solution (for definition see [1]).

Let $\mathcal{A}_0^*$ be the formal adjoint of $\mathcal{A}_0$, i.e. $\mathcal{A}_0^* \in L(V, V^*)$ is given by

$$\langle\mathcal{A}_0 v_1, v_2\rangle_{V^*,V} = \langle v_1, \mathcal{A}_0^* v_2\rangle_{V,V^*}, \quad \forall v_1, v_2 \in V,$$

and consider the following Cauchy problem

$$\begin{cases} \dfrac{dp}{dt}(t) = -\mathcal{A}_0^* p(t) + g(t), \quad t \in (0, T) \\ \\ p(T) = p_T. \end{cases} \tag{2.7}$$

Consider now the following stochastic differential equation

$$\begin{cases} dX(t) = F^0(X(t))dt + \sigma(X(t))dW(t), \quad t \in [0, T] \\ \\ X(0) = x \in \mathbb{R}^d \end{cases} \tag{2.8}$$

and the following related Kolmogorov equation

$$\begin{cases} \dfrac{\partial \phi}{\partial t}(t,x) = F^0(x)\cdot\nabla\phi(t,x) + \dfrac{1}{2}q_{ij}(x)\dfrac{\partial^2\phi}{\partial x_i\partial x_j}(t,x), & x\in\mathbb{R}^d,\ t\in(0,T) \\ \phi(0,x) = L^0(x) = \phi_0(x), & x\in\mathbb{R}^d. \end{cases} \tag{2.9}$$

If $F^0\in L^\infty(\mathbb{R}^d;\mathbb{R}^d)$, $L^0\in L^\infty(\mathbb{R}^d)$, then (2.9) has a unique weak solution.

Theorem 2.2.1. **([3])** *If $F^0:\mathbb{R}^d\longrightarrow\mathbb{R}^d$ is bounded and Lipschitz continuous, then there exists a unique solution X to (2.8). Moreover, if $L^0\in C_b(\mathbb{R}^d)$, then for any $t\in[0,T]$:*

$$\phi(t,x) = \mathbb{E}[L^0(X(t,x))] \qquad a.e.\ x\in\mathbb{R}^d, \tag{2.10}$$

where ϕ is the unique weak solution to (2.9).

Let us turn back to the stochastic optimal control problem (CP_S) and the related deterministic optimal control problem (CP). For any $u\in\mathcal{M}$ we define the functions $f^u:\mathbb{R}^d\longrightarrow\mathbb{R}^d$ and $a_u:V\times V\longrightarrow\mathbb{R}$, by $f^u(x)=f(x,u(x))$, $x\in\mathbb{R}^d$, $a_u(\varphi,\psi) = -\int_{\mathbb{R}^d} f^u\cdot\nabla\varphi\psi\rho\ dx + \frac{1}{2}\int_{\mathbb{R}^d}\frac{\partial\varphi}{\partial x_j}\frac{\partial}{\partial x_i}(q_{ij}\psi\rho)\ dx, \quad \forall\varphi,\psi\in V.$

Let φ_1^u the weak solution to (2.3), i.e. the weak solution to (2.9) corresponding to $F^0:=f^u$, $L^0:=G(\cdot,u(\cdot))$, and φ_2^u the weak solution to (2.4), i.e. the weak solution to (2.9) corresponding to $F^0:=f^u$, $L^0:=G_T$. It is obvious that for any $u\in\mathcal{M}_c$ we have that $F^0:=f^u$ (and so $a^0:=a_u$), $L^0:=G(\cdot,u(\cdot))$, and $L^0:=G_T$ satisfy the hypotheses in Theorem 2.2.1. Hence (2.1) has a unique solution X^u and

$$\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^d} G(X^u(t,x),u(X^u(t,x)))d\nu(x)\ dt\right] + \mathbb{E}\left[\int_{\mathbb{R}^d} G_T(X^u(T,x))d\nu(x)\right]$$
$$= \int_0^T\int_{\mathbb{R}^d}\varphi_1^u(t,x)d\nu(x)\ dt + \int_{\mathbb{R}^d}\varphi_2^u(T,x)d\nu(x).$$

It means that (CP_S) is equivalent to (CP_D).

Lemma 2.2.2. **([1])** *For any $v\in\mathcal{M}$, there exists $\{v^k\}_{k\in\mathbb{N}^*}\subset\mathcal{M}_c$ such that*

$$v^k\longrightarrow v \quad \text{in } H^m.$$

This means that $\mathcal{M}_c$ is a dense subset of $\mathcal{M}$ with respect to the distance of H^m.

By Lemma 2.2.2 we have that for any $u\in\mathcal{M}$ there exists a sequence $\{u_k\}_{k\in\mathbb{N}^*}\subset\mathcal{M}_c$ such that $u_k\longrightarrow u$ in H^m. As in the proof of Theorem 2.2.1 it follows that $\varphi_1^{u_k}\longrightarrow\varphi_1^u$, $\varphi_2^{u_k}\longrightarrow\varphi_2^u$, in $C([0,T];H)$. It follows that

$$\inf_{u\in\mathcal{M}}\left\{\int_0^T\int_{\mathbb{R}^d}\varphi_1^u(t,x)d\nu(x)\ dt + \int_{\mathbb{R}^d}\varphi_2^u(T,x)d\nu(x)\right\}$$
$$= \inf_{u\in\mathcal{M}_c}\left\{\int_0^T\int_{\mathbb{R}^d}\varphi_1^u(t,x)d\nu(x)\ dt + \int_{\mathbb{R}^d}\varphi_2^u(T,x)d\nu(x)\right\} = m^*.$$

We state now a result that will prove useful in what follows.

Lemma 2.2.3. **([3])** *For any $u\in\mathcal{M}$ and for any $h\in L^\infty(\mathbb{R}^d)$ such that $h(x)\geq 0$ a.e. $x\in\mathbb{R}^d$, the following problem*

$$\begin{cases} \dfrac{\partial \varphi}{\partial t}(t,x) = f^u(x)\cdot\nabla\varphi(t,x) + \dfrac{1}{2}q_{ij}(x)\dfrac{\partial^2\varphi}{\partial x_i\partial x_j}(t,x), & x\in\mathbb{R}^d,\ t\in(0,T) \\ \varphi(0,x) = h(x), & x\in\mathbb{R}^d \end{cases} \tag{2.11}$$

has a unique weak solution φ, and $\varphi(t,x) \geq 0$ a.e. $x \in \mathbb{R}^d$, for any $t \in [0,T]$.

2.3 Existence of an optimal control for the deterministic problem

The existence of an optimal control for problem (CP) will be proved under the additional assumption that $f(x,u) = f_0(x) + f_1(x)u$, where $f_0 : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $f_1 : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\cdot m}$ are Lipschitz continuous and bounded functions.

Assume that alongside hypotheses $(H2.1) - (H2.4)$ we have

(H2.5) For any $x \in \mathbb{R}^d$, the mapping $u \mapsto G(x,u)$ is convex on U_0 and $G|_{\mathbb{R}^d \times \tilde{U}_0} \in C_b^{0,1}(\mathbb{R}^d \times \tilde{U}_0)$.

Theorem 2.3.1. ([1, 3]) *There exists at least one optimal control u^* for problem (CP).*

2.4 The maximum principle for the deterministic optimal control problem

For any $u \in \mathcal{M}$ we consider the linear and continuous operator $\mathcal{A}_u : V \longrightarrow V^*$,

$$\mathcal{A}_u \varphi = f^u \cdot \nabla\varphi + \frac{1}{2} q_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}.$$

The formal adjoint of this operator, $\mathcal{A}_u^* : V \to V^*$ is given by

$$\mathcal{A}_u^* \psi = -\frac{1}{\rho} \nabla \cdot (f^u \psi \rho) + \frac{1}{2\rho} \cdot \frac{\partial^2}{\partial x_i \partial x_j}(q_{ij}\psi\rho), \quad \forall \psi \in V.$$

Assume that u^* is an optimal control for problem (CP). Let p_1^* be the unique weak solution (for definition see [1]) to the following backward equation

$$\begin{cases} \dfrac{dp_1}{dt}(t) = -\mathcal{A}_{u^*}^* p_1(t) + 1, \quad t \in (0,T) \\ \\ p_1(T) = 0, \end{cases} \tag{2.12}$$

i.e. p_1^* is the weak solution to

$$\begin{cases} \dfrac{\partial p_1}{\partial t}(t,x) = \dfrac{1}{\rho(x)} \nabla \cdot (f^{u^*}(x) p_1(t,x)\rho(x)) - \dfrac{1}{2\rho(x)} \cdot \dfrac{\partial^2}{\partial x_i \partial x_j}(q_{ij}(x) p_1(t,x)\rho(x)) + 1, & x \in \mathbb{R}^d,\ t \in (0,T) \\ \\ p_1(T,x) = 0, & x \in \mathbb{R}^d, \end{cases}$$

and let p_2^* be the unique weak solution to the next backward equation

$$\begin{cases} \dfrac{dp_2}{dt}(t) = -\mathcal{A}_{u^*}^* p_2(t), \quad t \in (0,T) \\ \\ p_2(T) = -1, \end{cases} \tag{2.13}$$

i.e. p_2^* is the weak solution to

$$\begin{cases} \dfrac{\partial p_2}{\partial t}(t,x) = \dfrac{1}{\rho(x)} \nabla \cdot (f^{u^*}(x) p_2(t,x)\rho(x)) - \dfrac{1}{2\rho(x)} \cdot \dfrac{\partial^2}{\partial x_i \partial x_j}(q_{ij}(x) p_2(t,x)\rho(x)), & x \in \mathbb{R}^d,\ t \in (0,T) \\ \\ p_2(T,x) = -1, & x \in \mathbb{R}^d. \end{cases}$$

Theorem 2.4.1. (The maximum principle) ([3]) *If G is independent of u and if u^* is an optimal control for problem (CP), then for almost any $x \in \mathbb{R}^d$:*

$$f(x, u^*(x)) \cdot \int_0^T \left[p_1^*(t,x) \nabla \varphi_1^{u^*}(t,x) + p_2^*(t,x) \nabla \varphi_2^{u^*}(t,x) \right] dt$$
$$= \max_{u_0 \in U_0} \left\{ f(x, u_0) \cdot \int_0^T \left[p_1^*(t,x) \nabla \varphi_1^{u^*}(t,x) + p_2^*(t,x) \nabla \varphi_2^{u^*}(t,x) \right] dt \right\}.$$

Remark 2.4.1. ([3]) If G does not depend on u and if in addition we have that $f|_{\mathbb{R}^d \times \tilde{U}_0} \in C_b^{0,1}(\mathbb{R}^d \times \tilde{U}_0; \mathbb{R}^d)$, then if u^* is an optimal control for problem (CP) we get that:

$$(D_u f^{u^*})^T(x) \int_0^T \left[p_1^*(t,x) \nabla \varphi_1^{u^*}(t,x) + p_2^*(t,x) \nabla \varphi_2^{u^*}(t,x) \right] dt \in N_{U_0}(u^*(x)) \subset \mathbb{R}^m,$$

a.e. $x \in \mathbb{R}^d$. We have denoted by $D_u f = \left(\frac{\partial f_i}{\partial u_l} \right)_{i=1,2,\ldots,d, l=1,2,\ldots,m}$ and $(D_u f)^T$ its transpose.

Assume now again that G depends both on x and u and that alongside $(H2.1) - (H2.4)$ we have that

(H2.2*) $f|_{\mathbb{R}^d \times \tilde{U}_0} \in C_b^{0,1}(\mathbb{R}^d \times \tilde{U}_0; \mathbb{R}^d)$;

(H2.4*) $G|_{\mathbb{R}^d \times \tilde{U}_0} \in C_b^{0,1}(\mathbb{R}^d \times \tilde{U}_0)$.

Theorem 2.4.2. (First order necessary optimality conditions) ([3]) *If u^* is an optimal control for problem (CP), then for a.e. $x \in \mathbb{R}^d$:*

$$p_1^*(0,x) \nabla_u G(x, u^*(x)) + (D_u f^{u^*})^T(x) \int_0^T [p_1^*(t,x) \nabla \varphi_1^{u^*}(t,x) + p_2^*(t,x) \nabla \varphi_2^{u^*}(t,x)] dt \in N_{U_0}(u^*(x)).$$

Remark 2.4.2. ([3]) In the particular case when $U_0 = \overline{B(0_m; \mu)}$ (where μ is a positive constant) we may conclude via Theorem 2.4.2 that for a.e. $x \in \mathbb{R}^d$:

$$u^*(x) \in \mu \; sign\{p_1^*(0,x) \nabla_u G(x, u^*(x)) + (D_u f^{u^*})^T(x) \int_0^T [p_1^*(t,x) \nabla \varphi_1^{u^*}(t,x) + p_2^*(t,x) \nabla \varphi_2^{u^*}(t,x)] dt\}.$$

Lemma 2.4.3. ([3]) *For any $u \in \mathcal{M}$, the weak solution p to*

$$\begin{cases} \dfrac{dp}{dt}(t) = -\mathcal{A}_u^* p(t) + 1, \quad t \in (0,T) \\ \\ p(T) = 0 \end{cases} \tag{2.14}$$

satisfies $p(0,x) < 0$ a.e. $x \in \mathbb{R}^d$.

Corollary 2.4.1. ([3]) If u^* is an optimal control for problem (CP), then by Theorem 2.4.2 and Lemma 2.4.3 we get that

$$u^*(x) \in (\nabla_u G(x, \cdot) + N_{U_0}(\cdot))^{-1}(F(x)) \quad \text{a.e. } x \in \mathbb{R}^d,$$

where $F(x) = -\dfrac{1}{p_1^*(0,x)} (D_u f^{u^*})^T(x) \displaystyle\int_0^T \left[p_1^*(t,x) \nabla \varphi_1^{u^*}(t,x) + p_2^*(t,x) \nabla \varphi_2^{u^*}(t,x) \right] dt.$

2.5 Examples and comments

Comment 2.5.1. **([3])** If one assumes that $\int_{\mathbb{R}^d} \rho(x)dx = 1$, then we may view ν as the distribution of $X^u(0)$ and $\mathbb{E}\left[G(X^u(t,x), u(X^u(t,x)))\right]$ and $\mathbb{E}\left[G_T(X^u(T,x))\right]$ as conditional expectations.

Example 2.5.1. **([3])** An important particular case is when $f(x,u) = f_0(x) + f_1(x)u$, with $f_0 \in C_b^1(\mathbb{R}^d;\mathbb{R}^d)$, $f_1 \in C_b^1(\mathbb{R}^d;\mathbb{R}^{d\cdot m})$, $G(x,u) = \frac{1}{2}|u|_m^2$ and $U_0 = \overline{B(0_m;\mu)}$, $\mu > 0$. By Theorem 2.4.2 we obtain that

$$p_1^*(0,x)u^*(x) + f_1(x)^T \int_0^T \left[p_1^*(t,x)\nabla\varphi_1^{u^*}(t,x) + p_2^*(t,x)\nabla\varphi_2^{u^*}(t,x)\right] dt \in N_{U_0}(u^*(x))$$

a.e. $x \in \mathbb{R}^d$. Since $p_1^*(0,x) < 0$ a.e. $x \in \mathbb{R}^d$ (see Lemma 2.4.3) we may conclude that

$$u^*(x) = P_{\overline{B(0_m;\mu)}}(\tilde{F}(x)) \quad \text{a.e. } x \in \mathbb{R}^d,$$

where $\tilde{F}(x) = -\dfrac{1}{p_1^*(0,x)} f_1(x)^T \displaystyle\int_0^T \left[p_1^*(t,x)\nabla\varphi_1^{u^*}(t,x) + p_2^*(t,x)\nabla\varphi_2^{u^*}(t,x)\right] dt.$

Example 2.5.2. **([3])** The case when $m = d$ and $f(x,u) = 1_{K_0}(x)u$ is the limit case for $f(x,u) = f_0(x) + f_1(x)u$, when $f_0 \equiv 0_d$, $f_1 = 1_{K_0}I_d$. Here K_0 is a closed and convex subset of $\mathbb{R}^d$ with $0_d \in K_0$. This case corresponds to the situation when the control acts only for $x \in K_0$. Function f does not fit hypothesis $(H2.2)$. However, the problem can be "approximated" by (CP_S) if we take $f_0 \equiv 0_d$ and $f_1 = 1_{K_0,\varepsilon}I_d$, where $1_{K_0,\varepsilon}$ is a "mollified" version of 1_{K_0} ($\varepsilon > 0$ is a small constant).

Remark 2.5.1. After a careful examination of Theorem 2.4.2, one obtains that, for any $u, v \in L^\infty(\mathbb{R}^d;\mathbb{R}^m)$ such that $u \in \mathcal{M}$ and $u + \varepsilon v \in \mathcal{M}$ for any $\varepsilon > 0$ sufficiently small, the directional derivative of I at u, for direction v is

$$dI(u)(v) = -\left\langle p_1^u(0)\nabla_u G^u + (D_u f)^T \int_0^T [p_1^u(t)\nabla\varphi_1^u(t) + p_2^u(t)\nabla\varphi_2^u(t)]dt, v \right\rangle_H,$$

where p_1^u, p_2^u are the weak solutions to (2.12) and (2.13), respectively, corresponding to $u^* := u$.

Based on this remark we propose a conceptual gradient type algorithm (see also [21], [22]) for problem (CP). The control u will be iteratively updated (improved).

An alternative approach

An alternative approach based on C_0-semigroups can be used to prove that problems (CP_S) and (CP_D) are equivalent and that the relationship between (CP_S) and (CP) holds if we view φ_1^u and φ_2^u as the mild solutions to (2.3) and (2.4), respectively.

2.6 Note on the considered approach

Remark 2.6.1. **([3])** An example of stochastic optimal control problem where expectations/conditional expectations are subject to some weighting J and J_T, respectively, is the following one

$(\mathbf{CP_S^1})$

$$\underset{u \in \mathcal{M}_c}{\text{Minimize}} \left\{ \int_0^T J\left(\int_{\mathbb{R}^d} \mathbb{E}[G(X^u(t,x), u(X^u(t,x)))]d\nu(x) \right) dt + J_T\left(\int_{\mathbb{R}^d} \mathbb{E}[G_T(X^u(T,x))]d\nu(x) \right) \right\}$$

(here $J, J_T \in C(\mathbb{R}^d)$). This problem is equivalent to the following deterministic one

$$(\mathbf{CP^1_D}) \qquad \underset{u\in\mathcal{M}_c}{\text{Minimize}} \left\{ \int_0^T J\left(\int_{\mathbb{R}^d} \varphi_1^u(t,x)d\nu(x)\right) dt + J_T\left(\int_{\mathbb{R}^d} \varphi_2^u(T,x)d\nu(x)\right)\right\},$$

and is deeply related to

$$(\mathbf{CP^1}) \qquad \underset{u\in\mathcal{M}}{\text{Minimize}} \left\{ \int_0^T J\left(\int_{\mathbb{R}^d} \varphi_1^u(t,x)d\nu(x)\right) dt + J_T\left(\int_{\mathbb{R}^d} \varphi_2^u(T,x)d\nu(x)\right)\right\}.$$

Under appropriate hypotheses on J and J_T, problem (CP^1) can be treated in an analogous manner to problem (CP).

The approach in this chapter can be extended to more general stochastic optimal control problems where the study cannot be reduced to a single Kolmogorov equation, as seen in Remark 2.6.1. Other such cases can be found in financial problems where the cost functional depends also on the variance of $G_T(X^u(T))$ (as a measure of the cost associated to the risk).

2.7 Auxiliary results

Lemma 2.7.1. **([1])** *$C_0^\infty(\mathbb{R}^d)$ is a dense subset of V.*

Lemma 2.7.2. **([1])** *If $\alpha \in V$, then $q_{ij}\dfrac{\partial^2\alpha}{\partial x_i \partial x_j} \in V^*$. Moreover, there exists a constant $\tilde{M} \geq 0$ such that*

$$\left\| q_{ij}\frac{\partial^2\alpha}{\partial x_i\partial x_j}\right\|_{V^*} \leq \tilde{M}\|\alpha\|_V, \quad \forall \alpha \in V.$$

3. OPTIMAL CONTROL OF STOCHASTIC DIFFERENTIAL EQUATIONS VIA FOKKER-PLANCK EQUATIONS

3.1 Formulation of the problem

Consider the following stochastic differential equation with feedback input

$$\begin{cases} dX(t) = f\left(t, X(t), u(t, X(t))\right) dt + \sigma(t, X(t))dW(t), \quad t \in [0,T], \\ \\ X(0) = X_0. \end{cases} \tag{3.1}$$

Here $T \in (0, +\infty)$, $d,\ n,\ m \in \mathbb{N}^*$, $W : [0,T] \times \Omega \to \mathbb{R}^n$ is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ with complete filtration, and X_0 is an $\mathbb{R}^d$-valued random variable which is independent of $(W(t))_{t\in[0,T]}$, and such that $\mathbb{E}[|X_0|_d^2] < +\infty$. Moreover, we assume that X_0 admits a probability density ρ_0.

Here $u(t, X)$ is a feedback controller (input) which belongs to the set of controllers

$$\mathcal{U} = \{v : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^m; \quad v \text{ is a Borel function}, v(t,x) \in U_0 \text{ a.e. } (t,x) \in [0,T] \times \mathbb{R}^d\},$$

where U_0 is a closed and bounded subset of $\mathbb{R}^m$ with $0_m \in U_0$. Notice that $u(t,x) = 0_m$ means that there is no input (action/control) at (t,x).

Unless stated otherwise, we assume throughout this chapter that the functions $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \longrightarrow \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d \cdot n}$, $q : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d \cdot d}$ have the following forms

$$f(t,x,u) = (f_1(t,x,u)\ f_2(t,x,u)\ ...\ f_d(t,x,u))^T, \quad \sigma(t,x) = (\sigma_{il}(t,x))_{i=1,2,...,d,\ l=1,2,...,n},$$
$$q(t,x) = \sigma(t,x)\sigma(t,x)^T = (q_{ij}(t,x))_{i,j=1,2,...,d} \quad (q_{ij}(t,x) = \sigma_{il}(t,x)\sigma_{jl}(t,x)),$$

for any $t \in [0,T], x \in \mathbb{R}^d, u \in \mathbb{R}^m$ and satisfy

(H3.1) $f|_{[0,T]\times\mathbb{R}^d\times U_0}$ is a bounded Borel function;

(H3.2) $\sigma \in C_b^1([0,T] \times \mathbb{R}^d; \mathbb{R}^{d \cdot n})$, and there exists a constant $\gamma > 0$ such that

$$\sigma(t,x)\sigma(t,x)^T y \cdot y = q_{ij}(t,x) y_i y_j \geq \gamma |y|_d^2, \quad \forall (t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d.$$

As concerns ρ_0 we assume for the sake of clarity of presentation that

(H3.3) $\rho_0 \in C_0(\mathbb{R}^d)$, $\rho_0(x) \geq 0$, $\forall x \in \mathbb{R}^d$, and $\displaystyle\int_{\mathbb{R}^d} \rho_0(x)dx = 1$.

We note that for any $u \in \mathcal{U}$, the following Fokker-Planck equation

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(t,x) = -\nabla \cdot (f^u(t,x)\rho(t,x)) + \dfrac{1}{2}\dfrac{\partial^2}{\partial x_i \partial x_j}(q_{ij}(t,x)\rho(t,x)), & t \in (0,T), x \in \mathbb{R}^d, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.2}$$

has a unique weak solution (to be defined in the next section) $\rho^u(t,x)$, which is a density function, i.e. $\forall t \in [0,T]$: $\rho^u(t,x) \geq 0$, a.e. $x \in \mathbb{R}^d$, $\displaystyle\int_{\mathbb{R}^d} \rho^u(t,x)dx = 1$ (see the next section).

Here and throughout this chapter one denotes $f^u(t,x) = f(t,x,u(t,x))$.

Under the above assumptions, we will deduce from the superposition principle that for any $u \in \mathcal{U}$ there exists a (unique in law/distribution) weak solution to (3.1) such that its density exists and is in fact $\rho^u(t)$, for any $t \in [0,T]$. We denote one of these solutions by X^u.

Assume that $G : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \longrightarrow \mathbb{R}$ and $G_T : \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfy

(H3.4) G is continuous with respect to $(t,x,u) \in [0,T] \times \mathbb{R}^d \times U_0$ and there exists $G_0 \in C([0,T] \times \mathbb{R}^d) \cap L^2((0,T) \times \mathbb{R}^d)$ such that $|G(t,x,u)| \leq G_0(t,x), \quad \forall (t,x,u) \in [0,T] \times \mathbb{R}^d \times U_0$;

(H3.5) $G_T \in C(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

For any $u \in \mathcal{U}$ one denotes $G^u(t,x) = G(t,x,u(t,x))$, and we get that

$$\mathbb{E}\left[\int_0^T G^u(t, X^u(t))dt\right] + \mathbb{E}[G_T(X^u(T))] = \int_0^T \int_{\mathbb{R}^d} G^u(t,x)\rho^u(t,x)dx\ dt + \int_{\mathbb{R}^d} G_T(x)\rho^u(T,x)dx.$$

This equality does not depend on the choice of X^u. Hence, we get that the stochastic optimal control problem with feedback inputs

$$(\mathbf{P_S}) \qquad \underset{u \in \mathcal{U}}{\text{Minimize}} \left\{ \mathbb{E}\left[\int_0^T G(t, X^u(t), u(t, X^u(t)))dt\right] + \mathbb{E}\left[G_T(X^u(T))\right] \right\}$$

is equivalent to the following deterministic optimal control problem with open-loop controllers

$$(\mathbf{P}) \qquad \underset{u \in \mathcal{U}}{\text{Minimize}} \left\{ \int_0^T \int_{\mathbb{R}^d} G^u(t,x)\rho^u(t,x)dx\ dt + \int_{\mathbb{R}^d} G_T(x)\rho^u(T,x)dx \right\}.$$

3.2 Weak solution to the Fokker-Planck equation. Relationship between the stochastic optimal control problem and a deterministic optimal control problem for a Fokker-Planck equation

Consider the following real Hilbert spaces: $V = H^1(\mathbb{R}^d)$ and $H = L^2(\mathbb{R}^d)$. We identify the dual of H with H and denote by $V^* = H^{-1}(\mathbb{R}^d)$ the dual of V, with the pairing $\langle\cdot,\cdot\rangle_{V,V^*}$ (or $\langle\cdot,\cdot\rangle_{V^*,V}$). Moreover, for any $\varphi \in V$, $\psi \in H$ we have $\langle\varphi,\psi\rangle_{V,V^*} = \langle\varphi,\psi\rangle_H$ (where $\langle\cdot,\cdot\rangle_H$ is the usual scalar product on H).

Note that the following embeddings $V \subset H \subset V^*$ are continuous and dense.

Let $F : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a bounded Borel function. Let us discuss now the relationship between the (probabilistically) weak solutions to the following stochastic differential equation (for definition see [20])

$$dX(t) = F(t, X(t))dt + \sigma(t, X(t))dW(t), \quad t \in [0,T], \tag{3.3}$$

the probability measure-valued solutions to the following Fokker-Planck equation (for definition see [20])

$$\frac{\partial}{\partial t}\mu(t) = L_t^*\mu(t), \quad t \in [0,T], \tag{3.4}$$

where $L_t(x) = F(t,x)\cdot\nabla + \dfrac{1}{2}q_{ij}(t,x)\dfrac{\partial^2}{\partial x_i \partial x_j}$ and L_t^* is its formal adjoint, and the weak solution to the following problem (for definition see [2])

$$\begin{cases} \dfrac{\partial\rho}{\partial t}(t,x) = -\nabla\cdot(F(t,x)\rho(t,x)) + \dfrac{1}{2}\dfrac{\partial^2}{\partial x_i\partial x_j}(q_{ij}(t,x)\rho(t,x)), & t \in (0,T), x \in \mathbb{R}^d, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{3.5}$$

Theorem 3.2.1. **([2])** *If $\rho_0 \in H$, there exists a unique weak solution ρ to* (3.5).

If in addition $\rho_0(x) \geq 0$ a.e. $x \in \mathbb{R}^d$, then for any $t \in [0,T]$, $\rho(t,x) \geq 0$ a.e. $x \in \mathbb{R}^d$.

Theorem 3.2.2. **([2])** *If ρ_0 satisfies (H3.3), then for any $t \in [0,T]$, $\rho(t,\cdot)$ is a probability density function, i.e.*

$$\rho(t,x) \geq 0, \text{ a.e. } x \in \mathbb{R}^d, \qquad \int_{\mathbb{R}^d}\rho(t,x)dx = 1.$$

Moreover, $\rho \in C([0,T];L^1(\mathbb{R}^d))$.

Remark 3.2.2. Since $\rho \in C([0,T];L^1(\mathbb{R}^d))$, it follows that for any $\psi \in C_b(\mathbb{R}^d)$, the mapping $t \mapsto \int_{\mathbb{R}^d}\psi(x)\rho(t,x)dx$ is continuous on $[0,T]$, i.e. if we consider $\nu : [0,T] \longrightarrow \mathcal{P}(\mathbb{R}^d)$ given by

$$\nu(t)(A) = \int_A \rho(t,x)dx, \quad \text{for any Borel set } A \subset \mathbb{R}^d,$$

then $\nu \in C([0,T];\mathcal{P}(\mathbb{R}^d))$.

If $(X(t))_{t\in[0,T]}$ is a weak solution to (3.3), then by Itô's formula we get that $(\mathcal{L}(X(t)))_{t\in[0,T]}$ is a probability measure-valued solution to (3.4).

The properties of ρ imply that $(\nu(t))_{t\in[0,T]}$ (as defined in Remark 3.2.2) is a probability measure-valued solution to (3.4) and satisfies that $\nu(0) = \mathcal{L}(X_0)$. Applying the superposition principle we may conclude that there exists a weak solution $(\tilde{X}(t))_{t\in[0,T]}$ to (3.3) such that $\mathcal{L}(\tilde{X}(t)) = \nu(t)$, $\forall t \in [0,T]$.

We turn now to the relationship with the weak solution to (3.5). By Remark 3.2.2 we have that $(\nu(t))_{t\in[0,T]}$ is a probability measure-valued solution to (3.4) satisfying that $\nu(0) = \mathcal{L}(X_0)$. It follows that there exists a weak solution $(X(t))_{t\in[0,T]}$ to (3.3) with $\mathcal{L}(X(0)) = \mathcal{L}(X_0)$, satisfying additionally $\mathcal{L}(X(t)) = \nu(t), \forall t \in [0,T]$.

If we assume in addition that F is uniformly Lipschitz continuous with respect to the variable x and since $\rho \in L^2((0,T)\times\mathbb{R}^d)$, then we get via theorem 1.3 in [28] that any probability measure-valued solution $(\mu(t))_{t\in[0,T]}$ to (3.4) with $\mu(0) = \nu(0)$ (has density ρ_0) satisfies $\mu(t) = \nu(t), \forall t \in [0,T]$ (this is a weak uniqueness result for (3.4) with initial value $\mathcal{L}(X_0)$). Consequently, any weak solution to (3.3) with initial law/distribution $\mathcal{L}(X_0)$ satisfies that its law is equal to $\nu(t), \forall t \in [0,T]$.

If, moreover, we assume in addition that F is continuous and uniformly Lipschitz continuous with respect to the variable x, then (3.3) has a unique strong solution $(X(t))_{t\in[0,T]}$, satisfying $X(0) = X_0$ (here we consider the filtration $\mathcal{F}_t$ generated by X_0 and $W(s)$, $s \in [0,t]$), which is also a weak solution to (3.3), and that in addition $\mathcal{L}(X(t)) = \nu(t)$, $\forall t \in [0,T]$ ($X(t)$ has $\rho(t)$ as a probability density).

Let us turn back to the relationship between (3.1) and (3.2), and between (P_S) and (P). Notice that for any $u \in \mathcal{U}$, $F := f^u$ is a bounded Borel function. Problem (3.2) has a unique weak solution ρ^u.

If for an arbitrary $u \in \mathcal{U}$ we denote by X^u any of the weak solutions to the SDE in (3.1) such that for any $t \in [0,T]$, $\mathcal{L}(X^u(t)) = \nu^u(t)$, where $\nu^u : [0,T] \longrightarrow \mathcal{P}(\mathbb{R}^d)$ is given by $\nu^u(t)(A) = \int_A \rho^u(t,x)dx$, for any Borel set $A \subset \mathbb{R}^d$ (i.e. $\rho^u(t)$ is a probability density for $\nu^u(t)$), then

$$\mathbb{E}\left[\int_0^T G^u(t, X^u(t))dt\right] + \mathbb{E}[G_T(X^u(T))] = I(u),$$

and we conclude that (P_S) and (P) are equivalent.

If $f|_{[0,T]\times\mathbb{R}^d\times U_0}$ is bounded, continuous and uniformly Lipschitz continuous with respect to (x,u), then for any $u \in \mathcal{U}_c = \mathcal{U} \cap C_b^{0,1}([0,T]\times\mathbb{R}^d;\mathbb{R}^m)$, $F := f^u$ is continuous and bounded, and Lipschitz continuous with respect to x. It is obvious that (3.1) has a unique strong solution (here we consider the filtration $\mathcal{F}_t$ generated by X_0 and $W(s)$, $s \in [0,t]$), that we may denote X^u. Indeed, it is also a weak solution to the SDE in (3.1) and satisfies that $\rho^u(t)$ is a probability density for $\mathcal{L}(X^u(t))$ for any $t \in [0,T]$.

We get that for any $u \in \mathcal{U}_c$: $\mathbb{E}\left[\int_0^T G^u(t, X^u(t))dt\right] + \mathbb{E}[G_T(X^u(T))] = I(u)$ and that

$$\inf_{u\in\mathcal{U}_c}\left\{\mathbb{E}\left[\int_0^T G(t, X^u(t), u(t, X^u(t)))dt\right] + \mathbb{E}\left[G_T(X^u(T))\right]\right\} = \inf_{u\in\mathcal{U}_c} I(u).$$

Notice that here X^u is actually the unique strong solution to (3.1).

Finally, if U_0 is also convex, then by Lemma 3.7.2 (in section 3.7) we get that $\mathcal{U}_c$ is a dense subset of $\mathcal{U}$ (with the topology of $L^2_{loc}([0,T]\times\mathbb{R}^d;\mathbb{R}^m)$). On the other hand, if additionally $f|_{[0,T]\times\mathbb{R}^d\times U_0}$ is bounded, continuous and uniformly Lipschitz continuous with respect to (x,u), then we have shown that

$$\inf_{u\in\mathcal{U}_c} I(u) = \inf_{u\in\mathcal{U}} I(u).$$

3.3 The maximum principle for the deterministic control problem

Assume in this section that f satisfies the stronger hypothesis

(H3.1') $f|_{[0,T]\times\mathbb{R}^d\times U_0}$ is a bounded and continuous function,

and that ρ_0 satisfies the weaker hypothesis

(H3.3') $\rho_0 \in L^2(\mathbb{R}^d)$, $\rho_0(x) \geq 0$ a.e. $x \in \mathbb{R}^d$, $\displaystyle\int_{\mathbb{R}^d} \rho_0(x)dx = 1$.

For any $u \in \mathcal{U}$ and $t \in [0,T]$ we define the linear operator $\mathcal{A}^u(t) : V \longrightarrow V^*$,

$$\mathcal{A}^u(t)\varphi = -\nabla \cdot (f^u(t,\cdot)\varphi) + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}(q_{ij}(t,\cdot)\varphi), \quad \forall \varphi \in V.$$

It follows in a standard way that $\mathcal{A}^u(t) \in L(V, V^*)$ and $\|\mathcal{A}^u(\cdot)\|_{L(V,V^*)} \in L^\infty(0,T)$.
Let $\mathcal{A}^u(t)^* \in L(V; V^*)$ be its formal adjoint, given by $\mathcal{A}^u(t)^*\psi = f^u(t,\cdot) \cdot \nabla\psi + \dfrac{1}{2} q_{ij}(t,\cdot)\dfrac{\partial^2 \psi}{\partial x_i \partial x_j}$.

Assume that u^* is an optimal control for problem (P). Let p be the unique weak solution (as defined below) to

$$\begin{cases} \dfrac{dp}{dt}(t) = -\mathcal{A}^{u^*}(t)^* p(t) + G(t,\cdot,u^*(t,\cdot)), & t \in (0,T), \\ \\ p(T) = -G_T, \end{cases} \tag{3.6}$$

This is a Cauchy problem in V^* and (3.6) might be equivalently written as

$$\begin{cases} \dfrac{\partial p}{\partial t} = -f^{u^*} \cdot \nabla p - \dfrac{1}{2} q_{ij} \dfrac{\partial^2 p}{\partial x_i \partial x_j} + G^{u^*}, & t \in (0,T), x \in \mathbb{R}^d, \\ \\ p(T,x) = -G_T(x), & x \in \mathbb{R}^d. \end{cases} \tag{3.7}$$

Theorem 3.3.1. (The maximum principle) ([2]) *If u^* is an optimal control for problem (P), then*

$$\rho^{u^*}(t,x)[f(t,x,u^*(t,x)) \cdot \nabla p(t,x) - G(t,x,u^*(t,x))] = \max_{u_0 \in U_0} \rho^{u^*}(t,x)[f(t,x,u_0) \cdot \nabla p(t,x) - G(t,x,u_0)],$$

a.e. $(t,x) \in (0,T) \times \mathbb{R}^d$.

Remark 3.3.1. ([2]) Actually, the maximum principle says that for almost any $(t,x) \in (0,T) \times \mathbb{R}^d$ we have that

$$u^*(t,x) = \arg\max\{\rho^{u^*}(t,x)[f(t,x,u_0) \cdot \nabla p(t,x) - G(t,x,u_0)];\ u_0 \in U_0\}.$$

If in addition we have that U_0 is convex and $f|_{[0,T]\times\mathbb{R}^d\times U_0} \in C_b^{0,0,1}([0,T]\times\mathbb{R}^d\times U_0;\mathbb{R}^d)$, $G|_{[0,T]\times\mathbb{R}^d\times U_0} \in C_b^{0,0,1}([0,T] \times \mathbb{R}^d \times U_0)$, then

$$\rho^{u^*}(t,x)((D_u f(t,x,u^*(t,x)))^T \nabla p(t,x) - \nabla_u G(t,x,u^*(t,x))) \in N_{U_0}(u^*(t,x)),$$

a.e. $(t,x) \in (0,T) \times \mathbb{R}^d$, where $D_u f = \left(\dfrac{\partial f_i}{\partial u_l}\right)_{i=1,2,\ldots,d,\ l=1,2,\ldots,m}$ and $(D_u f)^T$ is its transpose.

The time-independent case

If f, σ, G and G_0 are time-independent, then it is natural to consider the next stochastic optimal control problem with feedback inputs

$$(\mathbf{P_S^0}) \qquad \underset{u \in \mathcal{M}}{\text{Minimize}} \left\{ \mathbb{E}\left[\int_0^T G(X^u(t), u(X^u(t)))dt\right] + \mathbb{E}\left[G_T(X^u(T))\right] \right\},$$

and the following deterministic optimal control problem with open-loop controllers

$$(\mathbf{P^0}) \qquad \underset{u\in\mathcal{M}}{\text{Minimize}}\left\{\int_0^T\int_{\mathbb{R}^d} G(x,u(x))\rho^u(t,x)dx\ dt + \int_{\mathbb{R}^d} G_T(x)\rho^u(T,x)dx\right\},$$

where $\mathcal{M} = \{v : \mathbb{R}^d \longrightarrow \mathbb{R}^m;\ v \text{ is a Borel function},\ v(x) \in U_0,\ \text{a.e. } x \in \mathbb{R}^d\}$.

Theorem 3.3.3. **([2])** *If $u^* \in \mathcal{M}$ is an optimal control for problem (P^0), then*

$$\begin{aligned}
&f(x,u^*(x))\cdot\int_0^T \nabla p(t,x)\rho^{u^*}(t,x)dt - G(x,u^*(x))\int_0^T \rho^{u^*}(t,x)dt\\
&= \max_{u_0\in U_0}\left[f(x,u_0)\cdot\int_0^T \nabla p(t,x)\rho^{u^*}(t,x)dt - G(x,u_0)\int_0^T \rho^{u^*}(t,x)dt\right],
\end{aligned}$$

a.e. $x \in \mathbb{R}^d$, where p is the unique weak solution to (3.7).

Remark 3.3.2. **([2])** If in addition U_0 is convex and $f|_{\mathbb{R}^d\times U_0} \in C_b^{0,1}(\mathbb{R}^d \times U_0;\mathbb{R}^d)$, $G|_{\mathbb{R}^d\times U_0} \in C_b^{0,1}(\mathbb{R}^d \times U_0)$, then for a.e. $x \in \mathbb{R}^d$:

$$(D_u f(x,u^*(x)))^T \int_0^T \nabla p(t,x)\rho^{u^*}(t,x)dt - \nabla_u G(x,u^*(x))\int_0^T \rho^{u^*}(t,x)dt \in N_{U_0}(u^*(x)).$$

3.4 Existence of an optimal control for the deterministic control problem

Assume in this section that U_0 is also convex and that

$$f(t,x,u) = f_0(t,x) + f_1(t,x)u, \quad \forall(t,x,u) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^m,$$

where $f_0 : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$, $f_1 : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^{d\cdot m}$ are continuous and bounded. We also assume that besides $(H3.4)$ and $(H3.5)$, G and G_T satisfy that the mapping from U_0 to $\mathbb{R}$ given by

$$u \longmapsto G(t,x,u) \text{ is convex },\ \forall(t,x) \in [0,T]\times\mathbb{R}^d,$$

$$G(t,x,u) \geq \alpha_1, \quad \forall(t,x,u) \in [0,T]\times\mathbb{R}^d\times U_0, \quad G_T \in H^1(\mathbb{R}^d), \quad G_T(x) \geq \alpha_2, \quad \forall x \in \mathbb{R}^d.$$

Here α_1, α_2 are real constants.

Theorem 3.4.1. **([2])** *There exists at least one optimal control for (P).*

Remark 3.4.1. **([2])** Assume now that $f(x,u) = f_0(x) + f_1(x)u$, $\forall(x,u) \in \mathbb{R}^d\times\mathbb{R}^m$, where $f_0 : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $f_1 : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\cdot m}$ are continuous and bounded, and σ, G and G_0 are independent of t. It follows in a similar manner that there exists at least one optimal control for problem (P^0) (defined in section 3.3).

3.5 Examples

This section contains several examples; we recall one of them.

Example 3.5.1. **([2])** Assume that $U_0 = \overline{B(0_m;r)}$ $(r \in (0,+\infty))$, $f(t,x,u) = f_1(t,x)u$, where $f_1 : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^{d\cdot m}$ is continuous and bounded, $f_1(t,x) = 0_{d\cdot m}, \forall t \in [0,T], |x|_d \geq R$ $(R \in (0,+\infty))$,

$G(t,x,u) = \frac{1}{2}G_1(t,x)|u|_m^2$, where $G_1 \in C([0,T] \times \mathbb{R}^d), G_1(t,x) = 0, \forall t \in [0,T], |x|_d \geq R, G_1(t,x) > 0, \forall (t,x) \in (0,T) \times B(0_d; R)$. We obtain that

$$u^*(t,x) = P_{\overline{B(0_m;r)}}\left(\frac{f_1(t,x)^T \nabla p(t,x)}{G_1(t,x)}\right),$$

a.e. in $\{(t,x) \in (0,T) \times B(0_d; R); \rho^{u^*}(t,x) > 0\}$, where $P_{\overline{B(0_m;r)}}$ is the projection on $\overline{B(0_m, r)}$ and p is the weak solution to (3.7).

3.6 An optimal control problem with inputs with nonlocal action

Assume that the drift coefficient f is equal to u (i.e. $f(t,x,u) = u$) and that the control u does not explicitly depend on time. Hence, (3.1) becomes

$$\begin{cases} dX(t) = u(X(t))dt + \sigma(t, X(t))dW(t), \quad t \in [0,T], \\ \\ X(0) = X_0. \end{cases}$$

Function ρ, the weak solution to (3.2) may be viewed as the probabilistic density of a population.

Let $\zeta(x)$ be the density at $x \in \mathbb{R}^d$ of a second population (or of another entity) which produces a stimulus to the first population. For the sake of clarity, assume that the second population is time-independent, imobile, located in $\overline{B(0_d; R_0)}$ ($R_0 > 0$), and that it repels the individuals of the first population which are at a distance less than R (here R is a positive constant). It means that ζ is an input (control) with nonlocal action. This action is expressed mathematically in terms of the so-caled "generalized gradient" (nonlocal gradient) with kernel $\mathcal{G}_R$. Actually,

$$u(x) = -\nabla\left(\int_{\mathbb{R}^d} \mathcal{G}_R(x-y)\zeta(y)dy\right)$$

($u = -\nabla(\mathcal{G}_R * \zeta)$) describes the nonlocal action (effect) of the second population towards the individuals of the first population at $x \in \mathbb{R}^d$. We assume that function $\mathcal{G}_R$ is nonnegative, sufficiently smooth and its support is a subset of $\overline{B(0_d; R)}$.

The term $-\nabla \cdot (u(x)\rho(t,x)) = \nabla \cdot (\nabla(\mathcal{G}_R * \zeta)(x)\rho(t,x))$ in (3.2) describes a so-called cross-dispersion (see [12], [13]).

An appropriate set of controllers is

$$\mathcal{M}^0 = \{\zeta : \mathbb{R}^d \longrightarrow \mathbb{R};\ \zeta \text{ is a Borel function}, 0 \leq \zeta(x) \leq \tilde{M}_0 \text{ a.e. } x \in \mathbb{R}^d,\ \zeta(x) = 0 \text{ a.e. } |x|_d > R_0\}.$$

Here $\tilde{M}_0$ is a positive constant, and we assume that $\mathcal{G}_R \in C_0^2(\mathbb{R}^d)$, $\mathcal{G}_R(x) > 0$ if $|x|_d < R$ and $\mathcal{G}_R(x) = 0$ if $|x|_d \geq R$. The corresponding set of actions produced by the controllers ζ is

$$\mathcal{U}^0 = \left\{u;\ u(x) = -\nabla(\mathcal{G}_R * \zeta)(x),\ \forall x \in \mathbb{R}^d,\ \zeta \in \mathcal{M}^0\right\}.$$

Consider the following deterministic optimal control problem related to the controls $\zeta \in \mathcal{M}^0$:

$$(\mathbf{P^0_{nl}}) \qquad \underset{\zeta \in \mathcal{M}^0}{\text{Minimize}} \left\{\int_0^T \int_{\mathbb{R}^d} \overline{G}(x, \zeta(x))\rho^\zeta(t,x)dx\ dt + \int_{\mathbb{R}^d} G_T(x)\rho^\zeta(T,x)dx\right\}.$$

We use either the notation ρ^u or ρ^ζ (where $u(x) = -\nabla\left(\int_{\mathbb{R}^d} \mathcal{G}_R(x-y)\zeta(y)dy\right)$).

Assume that $\overline{G} \in C(\mathbb{R}^d \times [0, \tilde{M}_0])$ and there exists $\overline{G}_0 \in C(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that

$$|\overline{G}(x,\zeta)| \le \overline{G}_0(x), \quad \forall (x,\zeta) \in \mathbb{R}^d \times [0, \tilde{M}_0].$$

Moreover, we also assume that

$$\zeta \longmapsto \overline{G}(x,\zeta) \text{ is convex }, \ \forall x \in \mathbb{R}^d,$$

$$\overline{G}(x,\zeta) \ge \alpha_1, \quad \forall (x,\zeta) \in \mathbb{R}^d \times [0, \tilde{M}_0], \quad G_T(x) \ge \alpha_2, \quad \forall x \in \mathbb{R}^d.$$

Here α_1, α_2 are real constants.

The deterministic optimal control problem is obviously equivalent to the following one

$$(\mathbf{P^0_{Snl}}) \qquad \underset{\zeta \in \mathcal{M}^0}{\text{Minimize}} \left\{ \mathbb{E}\left[\int_0^T \overline{G}(X^\zeta(t), \zeta(X^\zeta(t)))dt \right] + \mathbb{E}\left[G_T(X^\zeta(T)) \right] \right\},$$

where we use either the notation X^u or X^ζ (where $u(x) = -\nabla \left(\int_{\mathbb{R}^d} \mathcal{G}_R(x-y)\zeta(y)dy \right)$).

Theorem 3.6.1. *There exists at least one optimal control ζ^* for problem (P^0_{nl}).*

Assume that in addition $\overline{G} \in C_b^{0,1}(\mathbb{R}^d \times [0, \tilde{M}_0])$.

Theorem 3.6.2. *If p is the weak solution to*

$$\begin{cases} \dfrac{\partial p}{\partial t} = \nabla(\mathcal{G}_R * \zeta^*) \cdot \nabla p - \dfrac{1}{2} q_{ij} \dfrac{\partial^2 p}{\partial x_i \partial x_j} + \overline{G}(\cdot, \zeta^*(\cdot)), & t \in (0,T), x \in \mathbb{R}^d, \\ p(T,x) = -G_T(x), & x \in \mathbb{R}^d, \end{cases}$$

then

$$\zeta^*(x) = \begin{cases} 0, & a.e. \ for \displaystyle\int_0^T \int_{\mathbb{R}^d} \rho^{\zeta^*}(t,y) \nabla p(t,y) \cdot \nabla \mathcal{G}_R(y-x) dy \ dt \\ & \quad + \dfrac{\partial \overline{G}}{\partial \zeta}(x, \zeta^*(x)) \displaystyle\int_0^T \rho^{\zeta^*}(t,x)dt > 0, \ |x|_d \le R_0 \\ \tilde{M}_0, & a.e. \ for \displaystyle\int_0^T \int_{\mathbb{R}^d} \rho^{\zeta^*}(t,y) \nabla p(t,y) \cdot \nabla \mathcal{G}_R(y-x) dy \ dt \\ & \quad + \dfrac{\partial \overline{G}}{\partial \zeta}(x, \zeta^*(x)) \displaystyle\int_0^T \rho^{\zeta^*}(t,x)dt < 0, \ |x|_d \le R_0. \end{cases}$$

3.7 Auxiliary results

Assume here that ρ_0 satisfies the weaker assumption $(H3.3')$ (instead of $(H3.3)$).

Lemma 3.7.1. ([2]) *There exists a nonnegative constant $\tilde{M}$ such that*

$$\|\rho^u\|_{C([0,T];H)} \le \tilde{M}, \quad \forall u \in \mathcal{U}.$$

Lemma 3.7.2. ([2]) *If U_0 is also convex, then for any $u \in \mathcal{U}$, there exists a sequence $\{u_k\}_{k \in \mathbb{N}^*} \subset \mathcal{U}_c$, such that*

$$u_k \longrightarrow u \quad \text{in } L^2_{loc}([0,T] \times \mathbb{R}^d; \mathbb{R}^m),$$

i.e. the closure of $\mathcal{U}_c$ in $L^2_{loc}([0,T] \times \mathbb{R}^d; \mathbb{R}^m)$ is $\mathcal{U}$.

Lemma 3.7.3. ([2]) *If $f|_{[0,T]\times\mathbb{R}^d\times U_0}$ is a continuous and bounded function and if $\{u_k\}_{k\in\mathbb{N}^*}\subset\mathcal{U}$ satisfies*

$$u_k \longrightarrow u, \quad \text{a.e. in } [0,T]\times\mathbb{R}^d,$$

then

$$\rho^{u_k} \longrightarrow \rho^u \quad \text{in } C([0,T];H).$$

4. FURTHER EXTENSIONS

4.1 Extensions to Chapter 2

A special attention will be paid to applications to real world problems modeled by stochastic differential equations (see [29] for SDEs in Physics and Engineering and [24] for Finance).

4.2 Extensions to Chapter 3

First extension. Consider the following stochastic optimal control problem with feedback inputs

$$(\overline{\mathbf{P}}_{\mathbf{S}}) \qquad \underset{u\in\mathcal{U}}{\text{Minimize}}\left\{\mathbb{E}\left[\int_0^T G^u(t,X^u(t))dt\right]+I_{\mathcal{K}_0}(X^u(T))\right\},$$

where $\mathcal{K}_0$ is a nonempty closed subset of $L^2(\Omega;\mathbb{R}^d)$. Notice that this is not a particular case of problem (P_S). We consider an "approximating" problem

$$(\overline{\mathbf{P}}^{\varepsilon}_{\mathbf{S}}) \qquad \underset{u\in\mathcal{U}}{\text{Minimize}}\left\{\mathbb{E}\left[\int_0^T G^u(t,X^u(t))dt\right]+\frac{1}{2\varepsilon}d_{\mathcal{K}_0}(X^u(T))^2\right\},$$

where $\varepsilon>0$ and $d_{\mathcal{K}_0}(y)=\inf\{\|y-z\|_{L^2(\Omega;\mathbb{R}^d)};\ z\in\mathcal{K}_0\}$.

For $\mathcal{K}_0=\{y\in L^2(\Omega;\mathbb{R}^d);\ \|y\|_{L^2(\Omega;\mathbb{R}^d)}\geq r_0\}$ $(r_0>0)$ we have that

$$d_{\mathcal{K}_0}(y)^2=(\min\{r_0,\|y\|_{L^2(\Omega;\mathbb{R}^d)}\}-r_0)^2.$$

Problem $(\overline{P}^{\varepsilon}_S)$ is equivalent to the following deterministic optimal control problem

$$(\overline{\mathbf{P}}^{\varepsilon}) \qquad \underset{u\in\mathcal{U}}{\text{Minimize}}\left\{\int_0^T\int_{\mathbb{R}^d}G^u\rho^u dx\, dt+\frac{1}{2\varepsilon}\left(\min\{r_0,(\int_{\mathbb{R}^d}|x|_d^2\rho^u(T,x)dx)^{\frac{1}{2}}\}-r_0\right)^2\right\}.$$

The methods used and developed in Chapter 3 could be adapted to investigate $(\overline{P}^{\varepsilon}_S)$ and $(\overline{P}^{\varepsilon})$ as well.
Second extension. Another possible continuation of the investigation in Chapter 3 concerns the deterministic optimal control problem (P) with ρ^u being, this time, the weak solution to the following Fokker-Planck equation with non-local term

$$\begin{cases}\dfrac{\partial\rho}{\partial t}(t,x)=-\nabla(f^u(t,x)\rho(t,x))+\dfrac{1}{2}\dfrac{\partial^2}{\partial x_i\partial x_j}(q_{ij}(t,x)\rho(t,x)) & \\ \qquad -\zeta(x)\rho(t,x)+\displaystyle\int_{\mathbb{R}^d}\zeta(y)\rho(t,y)\kappa(x,y)dy, & t\in(0,T),\ x\in\mathbb{R}^d, \\ \rho(0,x)=\rho_0(x), & x\in\mathbb{R}^d,\end{cases} \tag{4.1}$$

where $\zeta \in L^\infty(\mathbb{R}^d)$, $\zeta(x) \geq 0$ a.e. $x \in \mathbb{R}^d$, $\kappa \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, $\kappa(x,y) \geq 0$ a.e. $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$, $\displaystyle\int_{\mathbb{R}^d} \kappa(x,y)dx = 1$ a.e. $y \in \mathbb{R}^d$.
The probability density function corresponding to X^u satisfies (4.1) if, instead of just a simple Brownian noise, we consider the sum of two independent noises, a Brownian one and a Poisson type one (see [14], [11]).

4.3 Optimal control of a McKean-Vlasov equation via nonlinear Fokker-Planck equation

Consider the following deterministic optimal control problem

$$(\mathbf{P^1}) \qquad \underset{u \in \mathcal{U}^0}{\text{Minimize}} \int_0^T \int_{\mathbb{R}^d} G(t,x,u(x))\rho^u(t,x)dx\ dt + \int_{\mathbb{R}^d} G_T(x)\rho^u(T,x)dx,$$

where ρ^u is the "mild" solution, defined using the nonlinear semigroups, to the following nonlinear Fokker-Planck equation

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(t,x) = -\nabla \cdot (u(x)b(\rho(t,x))\rho(t,x)) + \Delta\beta(\rho(t,x)), & t \in (0,T),\ x \in \mathbb{R}^d \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Under appropriate hypotheses this problem is deeply related to the following stochastic optimal control problem (via the superposition principle)

$$(\mathbf{P^1_S}) \qquad \underset{\zeta \in \mathcal{M}^0}{\text{Minimize}}\ \mathbb{E}\left[\int_0^T G(t, X^u(t), \zeta(X^u(t)))dt\right] + \mathbb{E}[G_T(X^u(T))],$$

where X^u is a certain probabilistically weak solution to the following McKean-Vlasov equation

$$\begin{cases} dX(t) = u(X(t))b\left(\frac{d\mathcal{L}_{X(t)}}{dx}(X(t))\right)dt + \sigma\left(\frac{d\mathcal{L}_{X(t)}}{dx}(X(t))\right)dW(t), & t \in [0,T] \\ X(0) = X_0. \end{cases}$$

Here $\sigma(r) = \sqrt{2}\left(\frac{\beta(r)}{r}\right)^{\frac{1}{2}}$ and $u = -\nabla(\mathcal{G}_R * \zeta)$. The controllers $\zeta \in \mathcal{M}^0$ have nonlocal actions $u \in \mathcal{U}^0$ (as in section 3.6).

We intend to investigate problem (P^1) using an approach based on the nonlinear semigroups in $L^1(\mathbb{R}^d)$. The properties of the solutions to the nonlinear Fokker-Planck equation can be found in [6], [7], [8], [9]. In order to tackle the deterministic optimal control problem, the idea is to consider the backward Euler approximation of the nonlinear Fokker-Planck equation and the corresponding optimal control problem. The existence of an optimal control and the optimality conditions for this approximating problem are easier to derive than for problem (P^1). These ideas are being developed in [4].

5. **APPENDIX**

We recall here a few results that were indispensable throughout this PhD thesis:

Gronwall's inequality, Lions' existence and uniqueness theorem, Aubin's compactness theorem, some results concerning dissipative operators and C_0-semigroups.

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