1. Analytic brancches of eigenvalues and eigenforms , clustering
(Summary)
The main result of this course is the following theorem.

Theorem 0.1 *Suppoae M^n closed manifold equipped with a Morse function f with c_0 critical points of index 0, c_1 critical points of index $1, \cdots$ c_n critical points of index n, and a Riemannian metric g. Suppose that in the neighborhood of any critical point $y \in Cr(f)$ exists a chart $M \supset U_y,\ \varphi_y : (U_y, y) \to (\mathbb{R}^n, 0)$ s.t. $f \cdot \varphi_y^{-1} = f(y) - 1/2 \sum_{\leq k} x_i^2 + 1/2 \sum_{i \geq k+1} x_i^2$ and $(\varphi_y^{-1})^* g = \delta_{i,j}$. Then*

- For any q one has the countable collection $\mathcal{A}_q$ of **analytic functions in** *t, $\{\lambda^q_\alpha(t) \in \mathbb{R}$ and $\omega^q_\alpha(t) \in \Omega^q(M)\}$, such that*

1. $\Delta^f_q(t)\omega^q_\alpha(t) = \lambda^q_\alpha(t)\omega^q_\alpha(t)$,

2. $||\omega^q_\alpha(t)|| = 1$, $\omega^q_\alpha(t) \perp \omega^q_\beta(t)$ for $\alpha \neq \beta$.

3.$\lambda^q_\alpha(t)$ exhaust the eigenvalues with multiplicilty of Δ^f_q and $\omega^q_\alpha(t)$ provide a complete othonormal system in the Hilbert space completion of $\Omega^q(M)$.

- For a generic set of smooth functions f the branches $\lambda^q_\alpha(t)$ are simple (= of multiplicity one).

-The collection $\mathcal{A}_q$ has the following properties:

1. *exactly $\dim H^r(M : \mathbb{R})$ branches $\lambda^q_\alpha(t)$ are identically zero,*
2. *(Witten) exactly c_q branches $\lambda^q_\alpha(t)$ satisfy $\lambda^q_\alpha(t) \leq Ce^{-t}$ for t large enough,*
3. *For any $N \in \mathbb{Z}_{\geq 0}$ a precise number, depending on $c_0, c_1, \cdots, c_n$ and not on M, of branches $\lambda^q_\alpha(t)$ satisfy $\lim_{t\to\infty} \frac{\lambda^q_\alpha(t)}{t} = 2N$.*

As a consequence
$$\mathcal{A}_q = \mathcal{A}_q(0) \sqcup \mathcal{A}_q(1) \sqcup \mathcal{A}_q(2) \cdots \sqcup \mathcal{A}_q(N) \cdots \sqcup \mathcal{A}_q(\infty)$$
with $\mathcal{A}_q(N)$ finite and $\mathcal{A}_q(\infty)$ conjecturally $= \emptyset$.

Call $\mathcal{A}_q(N)$ the $N-$th cluster and $\mathcal{A}_q(0)$ the **virtually small spectral package**.

Define $(\Omega^q_N(M)(t), d_q(t))$ the subcomplex of $(\Omega^q(M), d_q(t))$ generated by the eigenforms $\omega^q_\alpha(t)$ with α corresponding to the eigenvalue branch with $\lim_{t\to\infty} \frac{\lambda^q_\alpha(t)}{t} = 2N$, equivalently the cluster $\mathcal{A}_q(N)$.

Consider the scaling $S(t) : (C^q, \partial_q) \to (C^q, \partial_q(t))$ defined by
$$S_q(t)(E_y) = (\pi/t)^{n-2q/4} e^{-tf(x)} E_y,$$
(for $y \in Cr_q(f),\ E_y \in Maps(Cr_q(f)\mathbb{R})$ with $E_y(y') := \delta_{y,y'}$)

In view of Helffer -Sjöstrand estimates the composition
$$(\Omega^q_0(M), d_q(t)) \xrightarrow{\subset} (\Omega^q_0(M), d_q(t)) \xrightarrow{e^{th}} \Omega^q(M), d_q(t)) \xrightarrow{Int} (C^q, \partial_q) \xrightarrow{S(t)} (C^q, \partial_q(t))$$
is an "asymptotical an isometry", i.e. equal to $I + O(1/t)$ with I an isometry (when $C^q = Maps(Cr_q, \mathbb{R})$ is equipped with the scalar product which makes E'_xs orthonormal).

The fact that the composition $Int \cdot e^{th} \cdot \subset$ is an isomorphism is referred to as Witten theorem.

Restrict to $t = 0$ and obtain the collection of finite dimensional sub complexes of $(\Omega^q(M)_N, d) \subset (\Omega^q(M), d)$ referred to as the $N-$th spectral package of the triple (M, g, f) The dimension of $\Omega^q(M)_N = \sharp\mathcal{A}_q(N)$. All of these complexes but for $N = 0$ are acyclic and the one for $N = 0$ is canonically isomorphic to the geometric complex hence carries the entire homological information about M plus more (to be studied).

About the proof of the theorem

For the Morse function f denote by $y \in Cr(f)$ a critical point and let $k(y)$ the Morse index of y.

We want to prove that either $\lim_{t\to\infty} \frac{\lambda^q_\alpha(t)}{t}$ is an even integer or the limit does not exists as a real number (an optimistic conjecture). Actually we want to show that for each $\epsilon > 0$ and for each N there exists C, T positive real numbers s.t. for $t > T$

$Spect\Delta^f_q(t) \subset \sqcup_N (2N - Ct^{-\epsilon}, 2N + Ce^{-Ct^\epsilon})$

Instead we will check that

$$Spect\Delta^f_q(t) \cap (2N + Ce^{-Ct^\epsilon}, 2N + 2 - Ct^{-\epsilon},) = \emptyset$$

by applying "the spectral gap" lemma with $a = 2N + Ce^{-Ct^\epsilon}$ and $b = (2N+2) - Ct^{-\epsilon}$.

For this purpose we will compare the formally self adjoint operator $\Delta^f_q(t) : \Omega^q(M) \to \Omega^q(M)$ with the formally self adjoint operator $D_q(t) : \Omega^q_S \to \Omega^q_S$, referred to as the Model operator for the Morse function f,

$D_q(t) := \oplus_{y\in C(f)} \Delta^{n,k(y)}_q(t)$, on the space equipoed with the scalar product induced by the metric $\delta_{i,j}$ on each copy $\mathbb{R}^n_y$ of $\mathbb{R}^n$.

$\Omega^q_S := \oplus_{y\in Cr(f)} \Omega^q_S(R^n_y) = \Omega^q_S(\sqcup_{y\in Cr(f)} \mathbb{R}^n_y)$ where R^n_y denotes a copy or R^n for each critical point y and

$\Delta^{n,k}_q(t)$ is the Witten Laplacian for the Riemannian manifold $\mathbb{R}^n$ equipped with the metric $\delta_{i,j}$ and the Morse function

$f_k(x_1, x_2, \cdots x_n) := -1/2 \sum_{i\le k} x_i^2 + 1/2 \sum_{i\ge k+1} x_i^2$.

This is what we refer to as the multidimensional (quantum) harmonic oscillator

Recall that $\Omega^q_S(\mathbb{R}^n)$ denotes the $q-$forms on $\mathbb{R}^n$ whose coefficients are rapidlly decaying functions in $n-$variables. which is a direct sum of spaces $\Omega^q_{S,I}(\mathbb{R}^n)$ indexed by symbols $I = \{1 \le i_1 < i_2 < \cdots < i_q \le n\}$. A form $\omega \in \Omega^q_{S,I}(\mathbb{R}^n)$ can be written unquely as $a(x_1, x_2, \cdots x_n) dx_{i_1} \wedge dx_{i_2} \wedge \cdots dx_{i_q}$ with $a(x_1, x_2, \cdots x_n) \in S$ and the operator $\Delta^{n,k}_q(t)$ acts componentwise on the functions $a(x_1, x_2, \cdots x_n)$ as

$$-\sum \partial^2/\partial x_i^2 + t^2 \sum x_i^2 + \epsilon(I)t$$

with $\epsilon(I) = \pm 1$, 1 if $k \in I$, and -1 otherwise.

The mathematics of the quantum harmonic oscillator implies that each eigenform of $D_q(t)$ is a direct sum of forms

$$e^{-t|x|^2/2} H_{r_1}(\sqrt{t}x_1) \cdot H_{r_2}(\sqrt{t}x_2) \cdots H_{r_n}(\sqrt{t}x_n)$$

with $H_r(y)$ are the Hermite polynomals $H_r(y) = y^r + \cdots$. and eigenvalues are constant in t and always even integers.

The comparison leads to the conclusion that the indexing $\mathcal{A}_q$ for $D_q(t)$ is a subset of the indexing for $\Delta^f_q(t)$ (it is conjecturally the same) and the limit of $\lambda^q_\alpha(t)/t$ when $t \to \infty$ are the same. for both operators when the case. To show this one proceeds as follows .

For any critical point y of f choose a Morse chart φ_y, $\varphi_y : (U_y, y) \to (R^n, 0)$, s.t. both the function f and the metric are in the standard form.

Choose $l > 0$ s.t the disc of radius l in $\mathbb{R}^n$ is contained in $\varphi(U_y)$ for any $y \in Cr(f)$ and a cutoff function $\chi : \mathbb{R}^n \to [0, \infty$ which is 1 for points in $x \in \mathbb{R}^n$ with $\sum x_i^2 \le l'$ and 0 when $\sum x_i^2 \ge l$ for $0 < l' < l$. Move using φ_y the forms $\tilde{\omega}^q_\alpha(x,t) = \chi(|x|)\omega^q_\alpha(x,t)$ for all α with $\lambda^q_\alpha = 2Nt$ into the forms with compact support $\underset{\sim}{\tilde{\omega}}^q_\alpha)$ on M.

Define H_1 the span of these forms and take H_2 the orthogonal complement in the Hilbert space completion of$\Omega^q(M)$. For t very large this choice satisfies the requirement of spectral gap lemma..

A complete proof of the above theorem will be soon available . Please follow arXive I a couple of months or contact me personally for the preprint in preparation.