VECTOR FIELDS, LYAPUNOV FUNCTION. MORSE THEORY
(basic definitions and results)

An $n \times n$ matrix A with real entries is called:

- **nondegenerate = nonsingular** if $\det A \neq 0$
- **hyperbolic** if all eigenvalues have real part $\neq 0$, in which case it has an **index** the number of eigenvalues with real part negative.
- A complete vector field $X \in \mathcal{X}(M)$ is equivalent to a **one parameter group of diffeomorphisms**,

$$\varphi : \mathbb{R} \times M \to M$$

  i.e. $\varphi(s+t,x) = \varphi(s,\varphi(t,x)), \text{and } \varphi(0,\mathrm{x}) = \mathrm{x}.$

  The map φ defines the derivation $X : \mathcal{C}(M) \to \mathcal{C}(M)$ by

$$X(f)(x) = \frac{d(f(\varphi(t,x))}{dt}_{t=0}$$

  equivalently $X(f)(\varphi(s,x_0)) = \frac{d(f(\varphi(t,x_0))}{dt}_{t=s}$ and a vector field $X = \sum_i a_i(x_1, x_2, \cdots x_n)\partial/\partial x_i$ in R^n defines $\varphi^i(t, x_1, x_2, \cdots x_n)$ as the solution of the ODE system

$$d\varphi(t,x)/dt = a(x), \;\; \varphi(0,x) = x \tag{1}$$

- The set of bf rest points $\mathcal{R}(X) := \{x \in M \mid \varphi(t,x) = x, \text{any t} \in \mathbb{R}\}$

  and for $x \in \mathcal{R}(x)$ define

$$W^{\pm}_x := \{y \in M \mid \lim_{t\to\pm\infty} \varphi(y,t) = x\}$$

  the **stable/ unstable set**.

- The vector field is **Hyperbolic** if the linearization of X at any rest point $x \in \mathbb{R}(X)$ is hyperbolic. If X is hyperbolic , the rest points are isolated and the stable set W^+_x and unstable sets W^-_x are smooth sub-manifolds diffeomorphic to $\mathbb{R}^{n-\text{index x}}$ resp. $\mathbb{R}^{\text{index x}}$.
- The hyperbolic vector field satisfies the **transversality condition** if for any $x, y \in \mathcal{R}(X)$ the unstable and stable submanifolds are transversal ($W^-_x \pitchfork W^+_y$). The transversality condition makes $W^-_x \cap W^+y$ a smooth submanifold of dimension (index x $-$ index y) and the space of trajectories from x to y, $\mathcal{T}(x,y)$, a smooth manifold of dimension $(\text{index x} - \text{index y} - 1)$.
- A smooth function $f : M \to \mathbb{R}$ is **Lyapunov** for X if $X(f)(x) < 0$ iff $x \neq \mathcal{R}(X)$.

  If X is hyperbolic, has smooth Lyapunov function and M is closed then $M = \bigcup_{x\in\mathbb{R}(X)} W^-_x$ (resp. $M = \bigcup_{x\in\mathbb{R}(X)} W^+_x$).

Theorem 0.1 *If X is a hyperbolic vector field on a closed smooth manifold M^n which admits a smooth Lyapunov function and satisfies the transversality condition then the collection of sets W^-_x are the interior of the cells of a smooth CW complex structure $\mathcal{M}$ [1]. on M and, in particular, integration theory via Stokes theorem induces $Int : (\Omega^*, d_*) \to (C^*(\mathcal{M}), \partial_*)$ a well defined map of chain complexes which by deRham theorem induces isomorphism in cohomology [2] .*

[1] defined up to the choice of orientations on W^-_x

[2] In the attached Appendix to Lecture 4 one provides the details for the proof of the above theorem under the additional hypothesis that f is Morse function and in the neighborhood of the rest points $X = -grad_g f$ with respect to a flat metric

Note that :

- for any hyperbolic vector field X with Lyapunov function f one can find arbitrarily (C^0-) closed vector field X' which agrees with X in a small neighborhood of the set $\mathcal{R}(X)$ and satisfies the transversality condition.

- for a smooth function $f : M \to \mathbb{R}$ with all critical points non-degenerated the set of Riemannian metrics g s.t. the vector field $-grad_g f$ is hyperbolic with the collection of unstable sets the interior of the cells of a smooth CW-complex structure is C^0-generic.

Recall that:

- A smooth manifold with corners$P$ is locally diffeomorphic to open sets in $\mathbb{R}^n_+ := \{(x, 1, x_2, \cdots x_n) \in \mathbb{R}^n \mid x_i \geq 0\}$. Denote by $\partial(\mathbb{R}^n_+) := (x, 1, x_2, \cdots x_n \in \mathbb{R}^n_+ \mid \prod x_i = 0\}$. The set of points in P which in some chart correspond to $\partial(\mathbb{R}^n_+)$ form a topological manifold of dimension $n-1$ which is smoothable, hence P is homeomorphic to a smooth manifold with boundary by a homeomorphism which is a diffeomorphism on the interior and restricts to a diffeomorphism on each face.

- The $k-$corner of P is the collection of points which in some and then any coordinate chart has exactly k coordinates equal to 0. They form a smooth manifold of dimension $n-k$. Each connected component of such corner is called face. If the interior of P is homeomorphic to R^n then by Poincaré conjecture (may be less) is homeomorphic to a disc.

The geometric complex

The CW complex structure on M has the cells indexed by the critical points of f, equivalently the rest points of X, but require specification of orientation for each unstable set. With such orientation specified one obtains a sign for each trajectory from the rest point x of index r to the rest point y of index $r-1$ and then take as the incidence number $\mathbb{I}_r(x, y)$ the sum of these $\pm 1's$. This is the geometric complex defined by the hyperbolic vector field X when it satisfies the transversality condition.

Here is a summary of the result in the attached paper (Attachment to lecture 4) which provides a smooth CW complex structure on M associated with the Morse pair (g, f).

First introduce the notation $x > y, x, y \in \mathcal{R}(X)$ provided $\mathrm{index\ x} > \mathrm{index\ y}$ and $W^-_X \cap W^+_y \neq \emptyset$, and, under the assumption of transversality condition satisfied, denote by

$$\partial_k \mathcal{B}(x) := \bigsqcup_{x > y_1 > y_2 > \cdots > y_k} \mathcal{T}(x, y_1) \times \mathcal{T}(y_1, y_2) \times \cdots \mathcal{T}(y_{k-1}, y_k) \times W^-_{y_k}.$$

which is a manifold of dimension $(\mathrm{index\ x} - \mathrm{k})$. Denote by $\hat{i}_k(x) : \partial_k \mathcal{B}(x) \to M$ the disjoint union of the projection of $\mathcal{T}(x, y_1) \times \mathcal{T}(y_1, y_2) \times \cdots \mathcal{T}(y_{k-1}, y_k) \times W^-_{y_k}$ onto $W^-_{y_k}$ followed by the inclusion $W^-_{y_k} \subset M$. Put $\partial_0 \mathcal{B}(x) := W^-_x$ and $\hat{i}_0(x)$ the inclusion $W^-_x \subset M$. With this notation the following theorem is proven in the attached material.

Theorem 0.2 *Assume that M is a smooth manifold with (h, X) a Morse Smale pair* [3] *and x a critical point. Then $\hat{W}^-_k = \bigsqcup_{k=0,1,\cdots} \partial_k \mathcal{B}(x)$ has a structure of compact manifold with corners with the $k-$corner equal $\partial_k \mathcal{B}(x)$ and the map $\hat{i}_x : \hat{W}^-_x \to M$ defined by $\hat{i}_k(x)$ on $\partial_k \mathcal{B}(x)$ is a smooth map.*

[3] for example $X = -grad_g f$