CALCULUS ON A CLOSED ORIENTED RIEMANNIAN MANIFOLD
(what you should know- a summary)

For M oriented smooth manifold a Riemannian metric g defines a $(M)-$linear operators $* :^q \Omega^r(M) \to \Omega^{n-r}(M)$ with the property

$$*^{n-r} \cdot *^r = (-1)^{r(n-r)} Id$$

and such that

$$(\omega, \omega') := \int \omega \wedge *^r, \omega',$$

$\omega, \omega' \in \Omega^r(M)$ is a scalar product.

Denote by :

1. $$d_r^* := (-1)^{nr+1} *^{n-r} \cdot d_{n-r-1} \cdot *^{r+1} : \Omega^{r+1}(M) \to \Omega^r(M)$$
   the formal adjoint of $d_r : \Omega^r(M) \to \Omega^{r+1}(M)$ (i.e. $(d_r\omega, \omega')(= (\omega, d_r^*\omega')$, $\omega \in \Omega^r(M), \omega' \in \Omega^{r+1}(M))$,

2. $$\Delta_r := d_r^* \cdot d_r + d_{r-1} \cdot d_{r-1}^*,$$
   $$d_r \cdot \Delta_r = \Delta_{r+1} \cdot d_r$$
   $$d_r^* \cdot \Delta_r = \Delta_{r-1} \cdot d_r^*$$

3. Define
   $$(\omega, \omega')_k := ((1 + \Delta_r)^{k/2}\omega, (1 + \Delta_r)^{k/2}\omega')$$

4. $$\Omega_-^r := img\ d_r^*,$$
   $$\Omega_+^r := img\ d_{r-1},$$
   $$\Delta_r^+ := d_{r-1} \cdot d_{r-1}^* : \Omega_+^r \to \Omega_+^r$$
   $$\Delta_r^- := d_r^* \cdot d_r : \Omega_-^r \to \Omega_-^r$$

5. $$\mathcal{H}^r := \ker d_r \cap \ker d_{r-1}^* = \ker \Delta_r.$$

Suppose (M, g) closed (i.e. compact) Riemannian manifold.

For a closed smooth manifold M $\Omega^r(M)$ can be equipped with the C_r-norms based on all derivatives up to order $\leq r$ and w.r. to all these norms it becomes a Frechet space A Riemannian metric induces in addition the norms generated by the scalar products $(\cdot, \cdot)_s$ (with $(\cdot, \cdot)_0 = (\cdot, \cdot)$). With respect of the collection of all C_r- norms is a Frechet space and with respect to the scalar products $(\cdot, \cdot)_s$ lead to Hilbert spaces $H_s(\Omega^r(M))$. with $H_0(\Omega^r(M))$ denote usually by $L_2^r(M)$, all having $\omega^r(M)$ as a dense subspace .

All operators d_r, d_r^* resp. Δ_r are continuous with respect to the Frechet topology but w.r to the topology induced from L_2^s for a specific s but only from H_s to $H_{s'}$ $s' < s$ resp. $s' < s - 1$.

With respect to L_2^s norms , on the Hilbert space $H_s(\Omega^r(M))$ the completion of $\Omega^r(M)$ w.r. to the scalar product, the operator Δ_r extends to a unique densely defined closed operator Δ_r which is self adjoint with spectrum a discrete countable collection of real numbers.

deRham - Hodge theorem

1. $H^r_{DM} := \ker d_r / \mathrm{i}mg d_{r-1} \simeq H^r(M;\mathbb{R})$ with $H^r(M;\mathbb{R})$ the singular cohomology.
2. $\Omega^r_+(M) := \mathrm{i}mg d_{r-1} \subset \Omega^r(M)$,

   $\Omega^r_-(M) := \mathrm{i}mg d^*_{r-1} \subset \Omega^r(M)$

   $\mathcal{H}^r(M) := \ker d_r \cap \ker \delta^*_r$ are closed subspaces w.r. to the Frechet topology , mutually orthogonal w.r. to all scalar products $(\cdot,\cdot)_s$ and satisfy

   $\Omega^r(M) = \Omega^r_+(M) \oplus \Omega^r_-(M) \oplus \mathcal{H}^r(M)$ with $d_r : \Omega^r_- \to \Omega^r_+$ bijective.
3. $\mathcal{H}^r(M) = \ker \Delta_r = H^r_{DM}(M) = H^r(M:\mathbb{R})$.

The subgroup or smooth integral forms in $\mathcal{H}^r(M)$ defines a Latice L_r isomorphic to $\mathcal{H}^r(M):\mathbb{Z})/Tor^r$ inside the finite dimensional Hilbert space $\mathcal{H}^r(M)$, hence a volume $V_r \in \mathbb{R}_{>0}$. Here Tor^r denotes the torsion subgroup of $H_r(M:\mathbb{Z})$.

Spectral Package of $(M.g)$denoted by $\mathcal{A}(M,g)$

Consists from the collection of eigenvalues with multiplicity and eigen-spaces (which are all finite dimensional subspaces in $\Omega^r(M)$) of the Laplace-Betrami operators Δ_r.

One denotes the spectral package associated with Δ_r by $\mathcal{A}_r$ which decomposes as $\mathcal{A}_r = \mathcal{A}^0_r \sqcup \mathcal{A}^-_r \sqcup \mathcal{A}^+_r$

Note that d_r identifies $\mathcal{A}^-_r$to $\mathcal{A}^+_{r+1}$ and $*^r$ identifies $\mathcal{A}_r$ to $\mathcal{A}_{n-r}$ identifying $\mathcal{A}^0_r$ to $\mathcal{A}^0_{n-r}$ and permuting the parts keeping the parts $\pm$ or permuting them depending on the parity of n and r.

Here $\mathcal{A}^0$ consists of the eigenvalue 0 (with its multiplicity and and its eigen-space), $\mathcal{A}^-$ resp. $\mathcal{A}^-$ consisting of eigenvalues of Δ^+ resp. Δ^- with their multiplicity and their eigen-spaces.