1. Implications of the Poincaré conjecture

We write $D^n \subseteq \mathbb{R}^n$ for the closed standard n-disk in Euclidean space, $S^{n-1} = \partial D^n$ for its boundary the $(n-1)$-sphere, and $B^n = D^n \setminus \partial D^n$ for its interior, the open n-ball. If true, everything below should be well known and we can jump to the final corollaries via citations.

Lemma. *Suppose W is a compact topological n-manifold with boundary ∂W whose interior $W \setminus \partial W$ is homeomorphic to $\mathbb{R}^n$. Then $W \setminus B^n$ is a compact topological h-cobordism between ∂W and S^{n-1}.*

Proof. We fix a homeomorphism $W \setminus \partial W \cong \mathbb{R}^n$ and use it to identify $W \setminus \partial W$ with Euclidean space $\mathbb{R}^n$. It is clear that $W \setminus B^n$ is a compact topological n-manifold with boundary $\partial W \sqcup S^{n-1}$. Using a tubular neighborhood of ∂W in $W \setminus B^n$, we see that the natural inclusion $\mathbb{R}^n \setminus B^n \to W \setminus B^n$ is a homotopy equivalence. It follows immediately that the natural inclusion $S^{n-1} \to W \setminus B^n$ is a homotopy equivalence too. It remains to show that the natural inclusion $\partial W \to W \setminus B^n$ is a homotopy equivalence as well. For this purpose let $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ denote the one parameter family of homeomorphisms given by $\phi_t(x) := e^t x$. Let $\varphi : \partial W \times [0,1) \xrightarrow{\cong} U$ be a tubular neighborhood of ∂W. More precisely, U is an open neighborhood of ∂W in $W \setminus B^n$ and φ is a homeomorphism onto U so that $\varphi_0 = \mathrm{id}_{\partial W}$. Choose $\varepsilon > 0$ so that $V := \varphi(\partial W \times (0, \varepsilon))$ satisfies $\phi_t(V) \subseteq U$, for all $t \geq 0$. Since the inclusion $\partial W \to U$ and $j : V \to U$ are both homotopy equivalences, it suffices to show that the natural inclusion $\iota : V \to \mathbb{R}^n \setminus B^n$ is a homotopy equivalence. Choose $t_0 \geq 0$ so that $\phi_{t_0}(\mathbb{R}^n \setminus B^n) \subseteq V$. We obtain a continuous map $\Phi := \phi_{t_0} : \mathbb{R}^n \setminus B^n \to V$. Clearly, ϕ_t, $0 \leq t \leq t_0$, provides a homotopy $\iota \circ \Phi \simeq \mathrm{id}_{\mathbb{R}^n \setminus B^n}$. Moreover, ϕ_t, $0 \leq t \leq t_0$, also provides a homotopy $j \circ \Phi \circ \iota \simeq j \circ \mathrm{id}_V$. Since $j : V \to U$ is a homotopy equivalence, we conclude that $\Phi \circ \iota \simeq \mathrm{id}_V$. Therefore $\Phi : \mathbb{R}^n \setminus B^n \to V$ is a homotopy inverse of $\iota : V \to \mathbb{R}^n \setminus B^n$, hence ι is a homotopy equivalence. This completes the proof. □

Corollary. *Suppose W is a compact topological n-manifold with boundary ∂W whose interior $W \setminus \partial W$ is homeomorphic to $\mathbb{R}^n$. Then W is homeomorphic to the disk D^n.*

Proof. In view of the lemma above ∂W is a homotopy sphere. According to the Poincaré conjecture it thus must be homeomorphic to the sphere. Attaching a collar to W, we obtain a compact topological manifold N which is homeomorphic to W and so that its interior $N \setminus \partial N$ contains a homeomorphic image of W. By extension of a homeomorphism $N \setminus \partial N \cong \mathbb{R}^n \cong S^n \setminus \{*\}$ we obtain a homeomorphism $\psi : N/\partial N \xrightarrow{\cong} S^n$. This homeomorphism ψ restricts to an embedding of ∂W into S^n. According to the Schönfliess theorem of Braun and Mazur, there exist a homeomorphism $h : (S^n, \psi(\partial W)) \to (S^n, S^{n-1})$ where S^{n-1} denotes the equator in S^n. Then $h \circ \psi$ maps W homeomorphically onto one hemisphere of S^n, hence W is homeomorphic to the disk D^n. □

Corollary. *$\hat{\pi}^- : \hat{W}^- \to \Sigma$ is a disk bundle, topologically.*

Corollary. *In the Morse case, the unstable manifolds provide a CW-decomposition.*