REPORT ON THE PAPER "CYCLIC COVERS AND TOROIDAL EMBEDDINGS"

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Cyclic covers are a useful tool in algebraic geometry. The simplest example is the field extension

$$K \subset K(\sqrt[n]{\varphi})$$

obtained by adjoining to a field the root of an element. For example, the equation $t^n - \varphi \prod_i z_i^{m_i}$ over K simplifies to $t^n - \prod_i z_i^{m_i}$ over $K(\sqrt[n]{\varphi})$.

In classical algebraic geometry, cyclic covers were used to construct new examples from known ones. Given a complex projective variety X, a torsion line bundle L over X induces canonically a cyclic (topological) covering $\pi \colon X' \to X$ such that $\pi^* L$ becomes trivial. If L has torsion index r and $s \in \Gamma(X, L^r)$ is nowhere zero, the covering can be constructed as the r-th root of s (as in the function field case, the pullback of s becomes an r-th power of a section of the pullback of L). We may denote it by $\pi \colon X[\sqrt[n]{s}] \to X$. The choice of s is not important, up to isomorphism, since X is compact. Many invariants of X' can be read off those of X, but with coefficients in negative powers of L. For example,

$$\pi_* \Omega^p_{X'} = \oplus_{i=0}^{r-1} \Omega^p_X \otimes L^{-i}.$$

So one may construct manifolds with prescribed invariants by taking roots of torsion line bundles on known manifolds. The process may be reversed: known statements on the invariants of X' translate into similar statements on X, twisted by negative powers of L. For example, the Kähler differential of X' decomposes into integrable flat connections on L^{-i}, so that $\oplus_{i=0}^{r-1} \Omega^\bullet_X(L^{-i})$ is the Hodge complex $\pi_* \Omega^\bullet_{X'}$ on X. In particular, the E_1-degeneration for $(\Omega^\bullet_{X'}, F)$ translates into the E_1-degeneration for $(\Omega^\bullet_{X'}(L^{-i}), F)$, for every i. This exchange of information between the total and base space of a cyclic cover is called the *cyclic covering trick*.

The range of applications of the cyclic covering trick extends dramatically if s is allowed to have zeros. In this case s is a non-zero global section of the n-th power of some line bundle L on X. The n-th root of s is defined just as above. We obtain for example the same formula

$$\pi_* \mathcal{O}_{X[\sqrt[n]{s}]} = \oplus_{i=0}^{n-1} L^{-i}.$$

The morphism π is still cyclic Galois and flat, but ramifies over the zero locus of s. The total space $X[\sqrt[n]{s}]$ may be disconnected (even if s vanishes nowhere), it may have several irreducible components, and it always has singularites over the zero locus of s. These singularities are partially resolved by the normalization $\bar{X}[\sqrt[n]{s}] \to X[\sqrt[n]{s}]$. The induced morphism $\bar{\pi} \colon \bar{X}[\sqrt[n]{s}] \to X$ is cyclic Galois and flat, and one computes

$$\bar{\pi}_* \mathcal{O}_{\bar{X}[\sqrt[n]{s}]} = \oplus_{i=0}^{n-1} L^{-i}(\lfloor \frac{i}{n} Z(s) \rfloor).$$

Here $Z(s)$ is the effective Cartier divisor cut out by s, and the round down of the $\mathbb{Q}$-divisor $\frac{i}{n} Z(s)$ is defined componentwise. If $\operatorname{Supp} Z(s)$ has no singularities, then $\bar{X}[\sqrt[n]{s}]$ has no singularities. Differential forms or vector fields on $\bar{X}[\sqrt[n]{s}]$ are computed in terms of X, L, and the $\mathbb{Q}$-divisor $\frac{1}{n} Z(s)$. For example

$$\bar{\pi}_* \Omega^p_{\bar{X}[\sqrt[n]{s}]} = \oplus_{i=0}^{n-1} \Omega^p_X(\log \operatorname{Supp}\{\frac{i}{n} Z(s)\}) \otimes L^{-i}(\lfloor \frac{i}{n} Z(s) \rfloor),$$

If the singularities of $\operatorname{Supp} Z(s)$ are at most simple normal crossing, then $\bar{X}[\sqrt[n]{s}]$ has at most quotient singularities, and if $Y \to \bar{X}[\sqrt[n]{s}]$ is a desingularization, with $\nu \colon Y \to X$ the induced generically finite morphism, then

$$\nu_* \Omega^p_Y = \oplus_{i=0}^{n-1} \Omega^p_X(\log \operatorname{Supp}\{\frac{i}{n} Z(s)\}) \otimes L^{-i}(\lfloor \frac{i}{n} Z(s) \rfloor).$$

This formula, and its logarithmic version, is behind the vanishing theorems used in birational classification (see [6, 9, 10]). Statements on divisors of the form $K_X + \sum_j b_j E_j + T$, with X nonsingular, $\sum_j E_j$ simple normal crossing, $b_j \in [0,1]$, and T a torsion $\mathbb{Q}$-divisor, are reduced to similar statements on Y with $b_j \in \{0,1\}$ and $T = 0$.

Cyclic covers also appear in semistable reduction [8]. In its simplest form, a complex projective family over the unit disc $f \colon \mathcal{X} \to \Delta$ has nonsingular general fibers $\mathcal{X}_t$ $(t \ne 0)$, while the special fiber $\mathcal{X}_0$ is locally cut out by monomials $\prod_{i=1}^d z_i^{m_i}$ $(m_i \in \mathbb{N})$ with respect to local coordinates $z_1, \ldots, z_d$. The family is semistable if moreover $\mathcal{X}_0$ is reduced. If we base change with $\sqrt[n]{t}$ (with n divisible by all multiplicities m_i), and normalize $\tilde{\mathcal{X}} \to \mathcal{X} \times_\Delta \tilde{\Delta}$, the new family $\tilde{\mathcal{X}} \to \tilde{\Delta}$ has reduced special fiber $\tilde{\mathcal{X}}_0$, and $\tilde{\mathcal{X}} \setminus \tilde{\mathcal{X}}_0 \subset \tilde{\mathcal{X}}$ is a *quasi-smooth toroidal embedding*. If the irreducible components of $\mathcal{X}_0$ are nonsingular, the toroidal embedding is also strict and $\tilde{\mathcal{X}}$ admits a combinatorial desingularization. An equivalent description of $\tilde{\mathcal{X}}$ is the normalization of the n-th root of f, viewed as a holomorphic function on $\mathcal{X}$. Therefore the local computations of [8] give in fact the following statement: if X is complex manifold, and $0 \ne s \in \Gamma(X, L^n)$ is such that $\Sigma = \operatorname{Supp} Z(s)$ is a normal crossing divisor, then $\bar{X}[\sqrt[n]{s}] \setminus \bar{\pi}^{-1}(\Sigma) \subset \bar{X}[\sqrt[n]{s}]$ is a quasi-smooth toroidal embedding, and $\bar{\pi}$ is a toroidal morphism.

Cyclic covers are used to classify the singularities that appear in the birational classification of complex manifolds. Such singularities $P \in X$ are normal, and the canonical Weil divisor K_X is a torsion element of $\operatorname{Cl}(\mathcal{O}_{X,P})$. If r is the torsion index, there exists a rational function $\varphi \in \mathbb{C}(X)^*$ such that $rK_X = \operatorname{div}(\varphi)$. The normalization of X in the Kummer extension $\mathbb{C}(X) \to \mathbb{C}(X)(\sqrt[r]{\varphi})$ becomes a cyclic cover $P' \in X' \overset{\pi}{\to} P \in X$. It is called the *index one cover* of $P \in X$, since being étale in codimension one, $K_{X'} = \pi^* K_X \sim 0$. The known method to classify $P \in X$ is to first classify the index one cover, and then understand all possible actions of cyclic groups (see [14]).

We have discussed roots of rational functions, (normalized) roots of multi sections of line bundles, and index one covers of torsion $\mathbb{Q}$-divisors on normal varieties. This paper gives a unified treatment of all these concepts, based on *normalized roots of rational functions on normal varieties*. One advantage is to remove the smoothness assumption and the use of desingularization in the cyclic covering trick for vanishing theorems, by working inside the category of toroidal embeddings.

To state the main result, let k be an algebraically closed field. Let X/k be an normal algebraic variety. Let φ be an invertible rational function on X, let n be a positive integer such that $\operatorname{char} k \nmid n$. Denote $D = \frac{1}{n} \operatorname{div}(\varphi)$, so that D is a $\mathbb{Q}$-Weil divisor on X with $nD \sim 0$. Let $\pi \colon Y \to X$ be the normalization of X with respect to the ring extension

$$k(X) \to \frac{k(X)[T]}{(T^n - \varphi)}.$$

The right hand side is a product of fields, and Y identifies with the disjoint union of the normalization of X in each field. By construction, Y/k is a normal algebraic variety (possibly disconnected).

a) The class of T becomes an invertible rational function ψ on Y such that $\psi^n = \pi^* \varphi$. We have $\pi^* D = \operatorname{div}(\psi)$ and

$$\pi_* \mathcal{O}_Y = \oplus_{i=0}^{n-1} \mathcal{O}_X(\lfloor iD \rfloor) \cdot \psi^i.$$

The morphism π is étale exactly over $X \setminus \operatorname{Supp}\{D\}$. It is flat if and only if the Weil divisors $\lfloor iD \rfloor$ $(0 < i < n)$ are Cartier.

b) Suppose $U \subseteq X$ is a quasi-smooth toroidal embedding and $D|_U$ has integer coefficients. Then $\pi^{-1}(U) \subseteq Y$ is a quasi-smooth toroidal embedding, and π is a toroidal morphism. Moreover, $\pi^*\tilde{\Omega}^p_{X/k}(\log \Sigma_X) \tilde{\to} \tilde{\Omega}^p_{Y/k}(\log \Sigma_Y)$ and by the projection formula

$$\pi_*\tilde{\Omega}^p_{Y/k}(\log \Sigma_Y) = \tilde{\Omega}^p_{X/k}(\log \Sigma_X) \otimes \pi_*\mathcal{O}_Y.$$

c) Suppose $\operatorname{char} k = 0$. Let $U \subseteq X$ and $U' \subseteq X'$ be toroidal embeddings over k, let $\mu \colon X' \to X$ be a proper morphism which induces an isomorphism $U' \tilde{\to} U$. Then

$$R^q\mu_*\tilde{\Omega}^p_{X'/k}(\log \Sigma_{X'}) = \begin{cases} \tilde{\Omega}^p_{X/k}(\log \Sigma_X) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

Statement a) is elementary. The toroidal part of b) is implicit in [8] if (X, Σ_X) is log smooth, as already mentioned. The general case (Theorem 3.8) is proved by reduction to the following fact: the normalized root of a toric variety with respect to a torus character consists of several isomorphic copies of a toric morphism (Proposition 3.7). The sheaf $\tilde{\Omega}^p_{X/k}(\log \Sigma_X)$ consists of the rational p-forms ω of X such that both $\omega, d\omega$ have at most simple poles, along the prime components of Σ_X. It is called the sheaf of logarithmic p-forms of $(X/k, \Sigma_X)$, in the sense of Zariski-Steenbrink. It is constructed by ignoring closed subsets of X of codimension at least two, so in general it is singular. But if $X \setminus \Sigma_X \subset X$ is a toroidal embedding, it is locally free [15, 3]. If X is nonsingular and Σ_X is a normal crossing divisor, this sheaf coincides with the sheaf of logarithmic forms $\Omega^p_{X/k}(\log \Sigma_X)$ in the sense of Deligne (see [6] for the algebraic version, with Σ_X assumed simple normal crossing). We note that differential forms or vector fields on Y can be computed without the toroidal assumption (Lemma 3.6).

Statement c) is the invariance of the logarithmic sheaves under different toroidal embeddings, which is interesting in itself. One corollary is that

$$H^q(X, \tilde{\Omega}^p_{X/k}(\log \Sigma_X)) \to H^q(X', \tilde{\Omega}^p_{X'/k}(\log \Sigma_{X'}))$$

is an isomorphism for every p, q. If X is proper and $(X, X \setminus U)$ is log smooth, the corollary follows from the E_1-degeneration of the spectral sequence induced in hypercohomology by the logarithmic De Rham complex endowed with the naive filtration (Deligne [4]). If X is projective and $X \setminus U$ is a simple normal crossing divisor, Esnault-Viehweg [5, Lemma 1.5] proved that the corollary implies c). We use the same idea, combined with a result of Bierstone-Milman [2], in order to compactify strict log smooth toroidal embeddings (Corollary 1.11).

The normalization of roots of multi sections of line bundles on normal varieties, and the index one covers torsion $\mathbb{Q}$-divisors on normal varieties, are both examples of normalized roots of rational functions. In practice, index one covers are most useful. They preserve irreducibility, so one can work in the classical setting of function fields. Their drawback is that they do not commute with base change to open subsets. For this reason, at least for proofs, we need to consider normalized roots of rational functions, which commute with étale base change.

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