Jeremy Tyson: Lipschitz, Hölder and Sobolev surjections from euclidean cubes.

Abstract:

We study highly regular Peano-type surjections from Euclidean cubes. When is a compact connected metric space the image of $[0,1]^n$ under a Lipschitz map? α-Holder map? $W^{1,p}$-Sobolev map?

Our main results read as follows:

Theorem 1. Every compact, quasiconvex, doubling metric space is the image of $[0,1]^n$ for any n under an α-Holder surjection, or under a Lipschitz surjection if n is sufficiently large.

Theorem 2. For each $n \geq 2$, every rectifiably connected metric space which is compact in its length metric is the image of $[0,1]^n$ under a metrically differentiable a.e. $W^{1,n}$-surjection.

The maps which we construct are rank one singular: they are already surjective when restricted to a suitable Cantor set C, and map the complement of C to a 1-rectifiable subset of the target.

The proof of Theorem 1 uses results of Lang and Schroeder on the Lipschitz extension problem for $CAT(k)$ targets. For each compact, quasiconvex metric space X we construct a compact metric tree, contained in a reflexive Banach space, which Lipschitz surjects to X. If X is in addition doubling, the tree can be chosen with Assouad dimension arbitrarily close to that of X.

As an application, we show that the first Heisenberg group, equipped with its Carnot-Carathéodory metric, is the Lipschitz image of $\mathbb{R}^n$ for each $n \geq 5$. We conclude with related results on the existence of highly regular surjections between Carnot groups.

This is joint work with Piotr Hajlasz.