Peter V. Dovbush: Normality and p-point sequences.

Abstract: Let X be a complex Banach manifold modelled on a complex Banach space of positive, possibly infinite, dimension; k_X denotes the infinitesimal Kobayashi pseudometric on X; K_X denotes the Kobayashi pseudodistance on X; and $\mathcal{O}(X)$ denotes the set of all holomorphic functions on X. Let s be the spherical metric on Riemann sphere which is associated with Riemann metric $ds(z, dz) = |dz|/(1 + |z|^2)$.

A family $\mathcal{G}$ of holomorphic functions in the open unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ is said to be normal if each sequence $\{g_j\} \subset \mathcal{G}$ has a subsequence which converges uniformly (with respect to the Euclidean metric) on compacta in Δ or diverges uniformly to ∞ on compacta in Δ.

A function f in $\mathcal{O}(X)$ is called normal if the family $\mathcal{F} = \{f \circ \varphi \mid \varphi \in \mathcal{O}(\Delta, X)\}$, where $O(\Delta, X)$ denotes the space of all holomorphic maps $\varphi : \Delta \to X$, forms a normal family.

A sequence $\{x_n\}$ of points in X is called a P-sequence of $f \in \mathcal{O}(X)$ if there is a sequence $\{y_n\}$ of points in X and a positive number $\epsilon > 0$ such that $K_X(x_n, y_n) \to 0$ as $n \to \infty$ but $s(f(x_n), f(y_n)) \geq \epsilon$ for each positive integer n.

Theorem 1. *Suppose X is a complex Banach manifold, and suppose k_X is a metric. The following statements are equivalent for $f \in \mathcal{O}(X)$.*
(a) *f is normal.*
(b) *There exists a constant $Q > 0$ such that*

$$Q_f(x) := \sup_{v \in T_x(X)\setminus\{0\}} \frac{ds(f(x), f_*(x)v)}{k_X(x, v)} < Q \text{ for all } x \in X.$$

(c) *There exists a constant $L > 0$ such that*

$$s(f(x), f(y)) \leq L \cdot K_X(x, y) \text{ for all } x, y \in X.$$

(d) *There is no P-sequence $\{x_n\}$ in X possessed by f.*

Theorem 2. *Let X and k_X be given as in Theorem 1, and $f \in \mathcal{O}(X)$. If $Q_f(x_m) \to \infty$ as $m \to \infty$, then $\{x_m\}$ contains a subsequence which is a P-sequence of f.*

The converse to Theorem 2 is not true in general, as the following example shows. Let $X = \Delta$, $f(z) = \exp \frac{i}{1-z} \in \mathcal{O}(\Delta)$, $z_n = \frac{n^2}{1+n^2} - \frac{i}{n+n^3}$, $w_n = \frac{n^2}{1+n^2}$. Here $Q_f(w_n) \to \infty$, and hence by Theorem 2 sequence $\{w_n\}$ contains a subsequence $\{w_m\}$ which is a P-sequence of f. Since $K_\Delta(z_m, w_m) \to 0$ a subsequence $\{z_m\} \subseteq \{z_n\}$ is a P-sequence of f too, while the sequence $Q_f(z_m) \to 0$ as $m \to \infty$.

Theorem 3. *Let X and k_X be given as in Theorem 1. A function $f \in \mathcal{O}(X)$ is a nonnormal function on X iff there exist sequences $\{x_m\}$ and $\{y_m\}$ of points in X, and a positive constant M such that $K_X(x_m, y_m) < M$ for all m, $\lim_{m\to\infty} f(x_m) = \infty$, and $\lim_{m\to\infty} f(y_m) = a \in \mathbb{C}$.*