Plane curves invariants from quantum braid groups representations

Cristian Anghel

"Simion Stoilow" Intitute of Mathematics

The Ninth Congress of Romanian Mathematicians
28 June - 03 July 2019, Galati, Romania

Abstract

- The aim of the talk is to present an idea of Libgober for constructing invariants of plane curves, starting with representation of the braid groups as input. For the later, we shall use the quantum representation, which in knot theory, gives the colored Jones polynomial and the general Lawrence representation. In the meantime, we shall overview also the notion of Zariski pairs, and a possible application of our method for detecting them. This is a joint work with Cristina Anghel and Martin Palmer.

OUTLINE

INTRODUCTION I

- The theory of knots and links in $\mathbb{S}^3$ has a long history and many invariants were introduced aiming to detect them.

- Among these, the Alexander and the famous Jones polynomials are the most well known, the last one being in fact only the first from a series indexed by natural numbers $N \geq 2$.

- In particular, the Alexander polynomial has many equivalent definitions, one of which start from the topology of the knot complement. Guided by co-dimension 2 analogy, people developed a similar formalism for the case of plane algebraic curves in the complex projective plane.

- The result, an Alexander type polynomial, has interesting properties and in particular is connected to the topology of the singularities of the curve.

INTRODUCTION I

- The theory of knots and links in $\mathbb{S}^3$ has a long history and many invariants were introduced aiming to detect them.

- Among these, the Alexander and the famous Jones polynomials are the most well known, the last one being in fact only the first from a series indexed by natural numbers $N \geq 2$.

- In particular, the Alexander polynomial has many equivalent definitions, one of which start from the topology of the knot complement. Guided by co-dimension 2 analogy, people developed a similar formalism for the case of plane algebraic curves in the complex projective plane.

- The result, an Alexander type polynomial, has interesting properties and in particular is connected to the topology of the singularities of the curve.

INTRODUCTION I

- The theory of knots and links in $\mathbb{S}^3$ has a long history and many invariants were introduced aiming to detect them.

- Among these, the Alexander and the famous Jones polynomials are the most well known, the last one being in fact only the first from a series indexed by natural numbers $N \geq 2$.

- In particular, the Alexander polynomial has many equivalent definitions, one of which start from the topology of the knot complement. Guided by co-dimension 2 analogy, people developed a similar formalism for the case of plane algebraic curves in the complex projective plane.

- The result, an Alexander type polynomial, has interesting properties and in particular is connected to the topology of the singularities of the curve.

Introduction I

- The theory of knots and links in $\mathbb{S}^3$ has a long history and many invariants were introduced aiming to detect them.

- Among these, the Alexander and the famous Jones polynomials are the most well known, the last one being in fact only the first from a series indexed by natural numbers $N \geq 2$.

- In particular, the Alexander polynomial has many equivalent definitions, one of which start from the topology of the knot complement. Guided by co-dimension 2 analogy, people developed a similar formalism for the case of plane algebraic curves in the complex projective plane.

- The result, an Alexander type polynomial, has interesting properties and in particular is connected to the topology of the singularities of the curve.

Introduction II

- In the first part of the talk I shall speak about the Alexander polynomial and its main properties.

- In the second part, we will see an ingenious idea of Libgober, which recover the Alexander polynomial via a general construction using braid group representations. For the particular case of what is called the reduced Burau representation, it is obtained the usual Alexander polynomial of the curve.

- The last part will be devoted to the applications of the Libgober method to a particular class of braid group representations, namely those which come from the world of quantum groups. In the knots/links case, this gives the coloured Jones polynomials. Also, we will discuss a strongly related family of representations obtained by Lawrence.

- We will end with a short discussion on Zariski pairs and about a possible application of the coloured Jones polynomial for detecting them.

Introduction II

- In the first part of the talk I shall speak about the Alexander polynomial and its main properties.

- In the second part, we will see an ingenious idea of Libgober, which recover the Alexander polynomial via a general construction using braid group representations. For the particular case of what is called the reduced Burau representation, it is obtained the usual Alexander polynomial of the curve.

- The last part will be devoted to the applications of the Libgober method to a particular class of braid group representations, namely those which come from the world of quantum groups. In the knots/links case, this gives the coloured Jones polynomials. Also, we will discuss a strongly related family of representations obtained by Lawrence.

- We will end with a short discussion on Zariski pairs and about a possible application of the coloured Jones polynomial for detecting them.

INTRODUCTION II

- In the first part of the talk I shall speak about the Alexander polynomial and its main properties.

- In the second part, we will see an ingenious idea of Libgober, which recover the Alexander polynomial via a general construction using braid group representations. For the particular case of what is called the reduced Burau representation, it is obtained the usual Alexander polynomial of the curve.

- The last part will be devoted to the applications of the Libgober method to a particular class of braid group representations, namely those which come from the world of quantum groups. In the knots/links case, this gives the coloured Jones polynomials. Also, we will discuss a strongly related family of representations obtained by Lawrence.

- We will end with a short discussion on Zariski pairs and about a possible application of the coloured Jones polynomial for detecting them.

INTRODUCTION II

- In the first part of the talk I shall speak about the Alexander polynomial and its main properties.

- In the second part, we will see an ingenious idea of Libgober, which recover the Alexander polynomial via a general construction using braid group representations. For the particular case of what is called the reduced Burau representation, it is obtained the usual Alexander polynomial of the curve.

- The last part will be devoted to the applications of the Libgober method to a particular class of braid group representations, namely those which come from the world of quantum groups. In the knots/links case, this gives the coloured Jones polynomials. Also, we will discuss a strongly related family of representations obtained by Lawrence.

- We will end with a short discussion on Zariski pairs and about a possible application of the coloured Jones polynomial for detecting them.

Plane curves and Alexander type invariants I

- In what follows we will fix a complex projective algebraic curve $C \subset \mathbb{CP}^2$.

- Alternatively we will be interested also in its affine part, denoted by abuse also with $C \subset \mathbb{C}^2$. In this case we will suppose that it is transverse to the line at infinity L_∞

- In this setting a fundamental question is the following:

QUESTION 1

To characterize (to find invariants for) the topological type of the pair $(\mathbb{C}^2, C)$ or $(\mathbb{CP}^2, C)$, or of the complement.

- Also, we can ask the following:

QUESTION 2

What is the dependence of the global invariants on the type of singularities of C, or on the relative position of the singularities.

Plane curves and Alexander type invariants I

- In what follows we will fix a complex projective algebraic curve $C \subset \mathbb{CP}^2$.
- Alternatively we will be interested also in its affine part, denoted by abuse also with $C \subset \mathbb{C}^2$. In this case we will suppose that it is transverse to the line at infinity L_∞
- In this setting a fundamental question is the following:

QUESTION 1

To characterize (to find invariants for) the topological type of the pair $(\mathbb{C}^2, C)$ or $(\mathbb{CP}^2, C)$, or of the complement.

- Also, we can ask the following:

QUESTION 2

What is the dependence of the global invariants on the type of singularities of C, or on the relative position of the singularities.

- In what follows we will fix a complex projective algebraic curve $C \subset \mathbb{CP}^2$.
- Alternatively we will be interested also in its affine part, denoted by abuse also with $C \subset \mathbb{C}^2$. In this case we will suppose that it is transverse to the line at infinity L_∞
- In this setting a fundamental question is the following:

QUESTION 1

To characterize (to find invariants for) the topological type of the pair $(\mathbb{C}^2, C)$ or $(\mathbb{CP}^2, C)$, or of the complement.

- Also, we can ask the following:

QUESTION 2

What is the dependence of the global invariants on the type of singularities of C, or on the relative position of the singularities.

Plane curves and Alexander type invariants I

- In what follows we will fix a complex projective algebraic curve $C \subset \mathbb{CP}^2$.
- Alternatively we will be interested also in its affine part, denoted by abuse also with $C \subset \mathbb{C}^2$. In this case we will suppose that it is transverse to the line at infinity L_∞
- In this setting a fundamental question is the following:

Question 1

To characterize (to find invariants for) the topological type of the pair $(\mathbb{C}^2, C)$ or $(\mathbb{CP}^2, C)$, or of the complement.

- Also, we can ask the following:

Question 2

What is the dependence of the global invariants on the type of singularities of C, or on the relative position of the singularities.

PLANE CURVES AND ALEXANDER TYPE INVARIANTS II

- Concerning the first question, one could remark that it is in complete analogy with the usual question in low dimensional topology for knots or links in $\mathbb{S}^3$.

- Also, the second one has a "bad" answer: as we will see, the global invariants depends not only on the type of the singularities, but also on their relative position.

- A famous example in this direction is the following due to Zariski:

ZARISKI'S SEXTICS

There are two degree 6 plane curves C_1 and C_2 with 6 cusps each as only singularities, such that their complements in $\mathbb{C}^2$ are not homeomorphic. In fact already the fundamental groups of the complements are non-isomorphic.

Plane curves and Alexander type invariants II

- Concerning the first question, one could remark that it is in complete analogy with the usual question in low dimensional topology for knots or links in $\mathbb{S}^3$.

- Also, the second one has a "bad" answer: as we will see, the global invariants depends not only on the type of the singularities, but also on their relative position.

- A famous example in this direction is the following due to Zariski:

Zariski's sextics

There are two degree 6 plane curves C_1 and C_2 with 6 cusps each as only singularities, such that their complements in $\mathbb{C}^2$ are not homeomorphic. In fact already the fundamental groups of the complements are non-isomorphic.

Plane curves and Alexander type invariants II

- Concerning the first question, one could remark that it is in complete analogy with the usual question in low dimensional topology for knots or links in $\mathbb{S}^3$.

- Also, the second one has a "bad" answer: as we will see, the global invariants depends not only on the type of the singularities, but also on their relative position.

- A famous example in this direction is the following due to Zariski:

Zariski's sextics

There are two degree 6 plane curves C_1 and C_2 with 6 cusps each as only singularities, such that their complements in $\mathbb{C}^2$ are not homeomorphic. In fact already the fundamental groups of the complements are non-isomorphic.

Plane curves and Alexander type invariants III

- It is interesting to point out that in the Zariski's example, what make in fact the difference is the fact that for C_1 the cusps are on a conic, while for C_2 not.

- The existence of the singularities is essential for the non-triviality of the problem. In the smooth case it is known that all pairs $(\mathbb{CP}^2, C)$ are equivalent and the $\pi_1(complement) \simeq \mathbb{Z}_d$, where $d = deg(C)$.

- Also, even in the presence of singularities and/or multiple components, the homology of the complement is a very weak invariant. In fact the following is known:

Homology of the complement

If $C = C_1 \cup C_2 \cup ... \cup C_r$ is the union of r irreducible components of degree $d_1, ..., d_r$, then
$$H_1(\mathbb{CP}^2 \setminus C) = \mathbb{Z}^{r-1} \oplus \mathbb{Z}_{(d_1,...,d_r)}.$$

Plane curves and Alexander type invariants III

- It is interesting to point out that in the Zariski's example, what make in fact the difference is the fact that for C_1 the cusps are on a conic, while for C_2 not.

- The existence of the singularities is essential for the non-triviality of the problem. In the smooth case it is known that all pairs $(\mathbb{CP}^2, C)$ are equivalent and the $\pi_1(\text{complement}) \simeq \mathbb{Z}_d$, where $d = \deg(C)$.

- Also, even in the presence of singularities and/or multiple components, the homology of the complement is a very weak invariant. In fact the following is known:

Homology of the complement

If $C = C_1 \cup C_2 \cup ... \cup C_r$ is the union of r irreducible components of degree $d_1, ..., d_r$, then
$$H_1(\mathbb{CP}^2 \setminus C) = \mathbb{Z}^{r-1} \oplus \mathbb{Z}_{(d_1,...,d_r)}.$$

Plane curves and Alexander type invariants III

- It is interesting to point out that in the Zariski's example, what make in fact the difference is the fact that for C_1 the cusps are on a conic, while for C_2 not.

- The existence of the singularities is essential for the non-triviality of the problem. In the smooth case it is known that all pairs $(\mathbb{CP}^2, C)$ are equivalent and the $\pi_1(complement) \simeq \mathbb{Z}_d$, where $d = deg(C)$.

- Also, even in the presence of singularities and/or multiple components, the homology of the complement is a very weak invariant. In fact the following is known:

Homology of the complement

If $C = C_1 \cup C_2 \cup ... \cup C_r$ is the union of r irreducible components of degree $d_1, ..., d_r$, then
$$H_1(\mathbb{CP}^2 \setminus C) = \mathbb{Z}^{r-1} \oplus \mathbb{Z}_{(d_1,...,d_r)}.$$

Plane curves and Alexander type invariants IV

- A much finer invariant is of course the fundamental group. Due to the following result, it is no big difference between the affine or the projective case:

Homotopy of the complement

We have the following central extension:

$$0 \to \mathbb{Z} \to \pi_1(\mathbb{C}^2 \setminus C) \to \pi_1(\mathbb{CP}^2 \setminus C) \to 0.$$

Moreover, the two fundamental groups have the same commutator.

- So we shall restrict to the affine case. The general method due to Zariski goes as follows:
- take a point $P \in L_\infty$, not on C and a generic projection from P $\pi : \mathbb{C}^2 \to L$ such that the critical fibers has either simple tangency in smooth points or if they pass through a singular point, they are transverse to the tangent cone of the singular point.

PLANE CURVES AND ALEXANDER TYPE INVARIANTS IV

- A much finer invariant is of course the fundamental group. Due to the following result, it is no big difference between the affine or the projective case:

HOMOTOPY OF THE COMPLEMENT

We have the following central extension:

$$0 \to \mathbb{Z} \to \pi_1(\mathbb{C}^2 \setminus C) \to \pi_1(\mathbb{CP}^2 \setminus C) \to 0.$$

Moreover, the two fundamental groups have the same commutator.

- So we shall restrict to the affine case. The general method due to Zariski goes as follows:
- take a point $P \in L_\infty$, not on C and a generic projection from P $\pi : \mathbb{C}^2 \to L$ such that the critical fibers has either simple tangency in smooth points or if they pass through a singular point, they are transverse to the tangent cone of the singular point.

Plane curves and Alexander type invariants IV

- A much finer invariant is of course the fundamental group. Due to the following result, it is no big difference between the affine or the projective case:

Homotopy of the complement

We have the following central extension:

$$0 \to \mathbb{Z} \to \pi_1(\mathbb{C}^2 \setminus C) \to \pi_1(\mathbb{CP}^2 \setminus C) \to 0.$$

Moreover, the two fundamental groups have the same commutator.

- So we shall restrict to the affine case. The general method due to Zariski goes as follows:
- take a point $P \in L_\infty$, not on C and a generic projection from P $\pi : \mathbb{C}^2 \to L$ such that the critical fibers has either simple tangency in smooth points or if they pass through a singular point, they are transverse to the tangent cone of the singular point.

PLANE CURVES AND ALEXANDER TYPE INVARIANTS V

- take a non-critical value $P_0 \in L$.
- The monodromy around critical values, gives a morphism from $\pi_1(L \setminus crit. \; val.)$ to the group of isotopy classes of homeomophisms of the fiber $\pi^{-1}(P_0)$ that fixes $\pi^{-1}(P_0) \cap C$.
- If r is the number of critical values and d is the degree of C, we obtained the following:

DEFINITION: BRAID MONODROMY

The above morphism $\theta : F_r \to B_d$ is called the braid monodromy of the pair (C, π).

- The braid monodromy is a finer invariant (depending on P_0 and π) than the fundamental group and in fact determines the later trough the following standard action of the braid group B_d on the free group F_d:

Plane curves and Alexander type invariants V

- take a non-critical value $P_0 \in L$.
- The monodromy around critical values, gives a morphism from $\pi_1(L \setminus crit.\ val.)$ to the group of isotopy classes of homeomophisms of the fiber $\pi^{-1}(P_0)$ that fixes $\pi^{-1}(P_0) \cap C$.
- If r is the number of critical values and d is the degree of C, we obtained the following:

Definition: Braid monodromy

The above morphism $\theta : F_r \to B_d$ is called the braid monodromy of the pair (C, π).

- The braid monodromy is a finer invariant (depending on P_0 and π) than the fundamental group and in fact determines the later trough the following standard action of the braid group B_d on the free group F_d:

PLANE CURVES AND ALEXANDER TYPE INVARIANTS V

- take a non-critical value $P_0 \in L$.
- The monodromy around critical values, gives a morphism from $\pi_1(L \setminus crit. \, val.)$ to the group of isotopy classes of homeomophisms of the fiber $\pi^{-1}(P_0)$ that fixes $\pi^{-1}(P_0) \cap C$.
- If r is the number of critical values and d is the degree of C, we obtained the following:

DEFINITION: BRAID MONODROMY

The above morphism $\theta : F_r \to B_d$ is called the braid monodromy of the pair (C, π).

- The braid monodromy is a finer invariant (depending on P_0 and π) than the fundamental group and in fact determines the later trough the following standard action of the braid group B_d on the free group F_d:

PLANE CURVES AND ALEXANDER TYPE INVARIANTS V

- take a non-critical value $P_0 \in L$.
- The monodromy around critical values, gives a morphism from $\pi_1(L \setminus crit.\ val.)$ to the group of isotopy classes of homeomophisms of the fiber $\pi^{-1}(P_0)$ that fixes $\pi^{-1}(P_0) \cap C$.
- If r is the number of critical values and d is the degree of C, we obtained the following:

DEFINITION: BRAID MONODROMY

The above morphism $\theta : F_r \to B_d$ is called the braid monodromy of the pair (C, π).

- The braid monodromy is a finer invariant (depending on P_0 and π) than the fundamental group and in fact determines the later trough the following standard action of the braid group B_d on the free group F_d:

Plane curves and Alexander type invariants VI

- we denote by σ_j the $d-1$ generators of B_d and by μ_i the d generators of F_d,
- the action $\sigma_j(\mu_i) =$
- μ_{i+1} if $i = j$
- $\mu_{i+1}\mu_i\mu_{i+1}^{-1}$ if $i = j + 1$
- μ_i else.
- With the above notations the following celebrated result of Zariski - van Kampen express the fundamental group of the complement in terms of the braid monodromy θ:

Zariski - van Kampen Theorem

$\pi_1(\mathbb{C}^2 \setminus C)$ is generated by the μ_i with relations $\theta(\alpha_j)\mu_i = \mu_i$ for $i = 1...d$, $j = 1...r$ and α_j the generators of the free group F_r.

PLANE CURVES AND ALEXANDER TYPE INVARIANTS VI

- we denote by σ_j the $d-1$ generators of B_d and by μ_i the d generators of F_d,
- the action $\sigma_j(\mu_i) =$
- μ_{i+1} if $i = j$
- $\mu_{i+1}\mu_i\mu_{i+1}^{-1}$ if $i = j+1$
- μ_i else.
- With the above notations the following celebrated result of Zariski - van Kampen express the fundamental group of the complement in terms of the braid monodromy θ:

ZARISKI - VAN KAMPEN THEOREM

$\pi_1(\mathbb{C}^2 \setminus C)$ is generated by the μ_i with relations $\theta(\alpha_j)\mu_i = \mu_i$ for $i = 1...d$, $j = 1...r$ and α_j the generators of the free group F_r.

Plane curves and Alexander type invariants VI

- we denote by σ_j the $d-1$ generators of B_d and by μ_i the d generators of F_d,

- the action $\sigma_j(\mu_i) =$

- μ_{i+1} if $i = j$

- $\mu_{i+1}\mu_i\mu_{i+1}^{-1}$ if $i = j+1$

- μ_i else.

- With the above notations the following celebrated result of Zariski - van Kampen express the fundamental group of the complement in terms of the braid monodromy θ:

Zariski - van Kampen Theorem

$\pi_1(\mathbb{C}^2 \setminus C)$ is generated by the μ_i with relations $\theta(\alpha_j)\mu_i = \mu_i$ for $i = 1...d$, $j = 1...r$ and α_j the generators of the free group F_r.

Plane curves and Alexander type invariants VI

- we denote by σ_j the $d-1$ generators of B_d and by μ_i the d generators of F_d,

- the action $\sigma_j(\mu_i) =$

- μ_{i+1} if $i = j$

- $\mu_{i+1}\mu_i\mu_{i+1}^{-1}$ if $i = j + 1$

- μ_i else.

- With the above notations the following celebrated result of Zariski - van Kampen express the fundamental group of the complement in terms of the braid monodromy θ:

Zariski - van Kampen Theorem

$\pi_1(\mathbb{C}^2 \setminus C)$ is generated by the μ_i with relations $\theta(\alpha_j)\mu_i = \mu_i$ for $i = 1...d$, $j = 1...r$ and α_j the generators of the free group F_r.

PLANE CURVES AND ALEXANDER TYPE INVARIANTS VI

- we denote by σ_j the $d - 1$ generators of B_d and by μ_i the d generators of F_d,
- the action $\sigma_j(\mu_i) =$
- μ_{i+1} if $i = j$
- $\mu_{i+1}\mu_i\mu_{i+1}^{-1}$ if $i = j + 1$
- μ_i else.
- With the above notations the following celebrated result of Zariski - van Kampen express the fundamental group of the complement in terms of the braid monodromy θ:

ZARISKI - VAN KAMPEN THEOREM

$\pi_1(\mathbb{C}^2 \setminus C)$ is generated by the μ_i with relations $\theta(\alpha_j)\mu_i = \mu_i$ for $i = 1...d$, $j = 1...r$ and α_j the generators of the free group F_r.

Plane curves and Alexander type invariants VI

- we denote by σ_j the $d-1$ generators of B_d and by μ_i the d generators of F_d,
- the action $\sigma_j(\mu_i) =$
- μ_{i+1} if $i = j$
- $\mu_{i+1}\mu_i\mu_{i+1}^{-1}$ if $i = j + 1$
- μ_i else.
- With the above notations the following celebrated result of Zariski - van Kampen express the fundamental group of the complement in terms of the braid monodromy θ:

Zariski - van Kampen Theorem

$\pi_1(\mathbb{C}^2 \setminus C)$ is generated by the μ_i with relations $\theta(\alpha_j)\mu_i = \mu_i$ for $i = 1...d$, $j = 1...r$ and α_j the generators of the free group F_r.

Plane curves and Alexander type invariants VII

- Even with the above explicit description, it is hard to decide when two particular curves have isomorphic fundamental groups.

- The idea is to extract less finer but more computable invariants. The Alexander polynomial for this situation is constructed as follows:

- one starts with the epi-morphism

$$\pi_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}^{r-1} \oplus \mathbb{Z}_{(d_1,\ldots,d_r)} \to \mathbb{Z},$$

- and one considers the usual associated cyclic cover $M \to \mathbb{C}^2 \setminus C$.

- Then, for any field R, $H_1(M, R)$ is an $R[t, t^{-1}]$-module of the following form:

$$\oplus \frac{R[t, t^{-1}]}{(\lambda_i)}.$$

Plane curves and Alexander type invariants VII

- Even with the above explicit description, it is hard to decide when two particular curves have isomorphic fundamental groups.
- The idea is to extract less finer but more computable invariants. The Alexander polynomial for this situation is constructed as follows:
- one starts with the epi-morphism

$$\pi_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}^{r-1} \oplus \mathbb{Z}_{(d_1,\ldots,d_r)} \to \mathbb{Z},$$

- and one considers the usual associated cyclic cover $M \to \mathbb{C}^2 \setminus C$.
- Then, for any field R, $H_1(M, R)$ is an $R[t, t^{-1}]$-module of the following form:

$$\oplus \frac{R[t, t^{-1}]}{(\lambda_i)}.$$

PLANE CURVES AND ALEXANDER TYPE INVARIANTS VII

- Even with the above explicit description, it is hard to decide when two particular curves have isomorphic fundamental groups.
- The idea is to extract less finer but more computable invariants. The Alexander polynomial for this situation is constructed as follows:
- one starts with the epi-morphism

$$\pi_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}^{r-1} \oplus \mathbb{Z}_{(d_1,\ldots,d_r)} \to \mathbb{Z},$$

- and one considers the usual associated cyclic cover $M \to \mathbb{C}^2 \setminus C$.
- Then, for any field R, $H_1(M, R)$ is an $R[t, t^{-1}]$-module of the following form:

$$\oplus \frac{R[t, t^{-1}]}{(\lambda_i)}.$$

Plane curves and Alexander type invariants VII

- Even with the above explicit description, it is hard to decide when two particular curves have isomorphic fundamental groups.
- The idea is to extract less finer but more computable invariants. The Alexander polynomial for this situation is constructed as follows:
- one starts with the epi-morphism

$$\pi_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}^{r-1} \oplus \mathbb{Z}_{(d_1,\ldots,d_r)} \to \mathbb{Z},$$

- and one considers the usual associated cyclic cover $M \to \mathbb{C}^2 \setminus C$.
- Then, for any field R, $H_1(M, R)$ is an $R[t, t^{-1}]$-module of the following form:

$$\oplus \frac{R[t, t^{-1}]}{(\lambda_i)}.$$

Plane curves and Alexander type invariants VII

- Even with the above explicit description, it is hard to decide when two particular curves have isomorphic fundamental groups.
- The idea is to extract less finer but more computable invariants. The Alexander polynomial for this situation is constructed as follows:
- one starts with the epi-morphism

$$\pi_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}^{r-1} \oplus \mathbb{Z}_{(d_1,\ldots,d_r)} \to \mathbb{Z},$$

- and one considers the usual associated cyclic cover $M \to \mathbb{C}^2 \setminus C$.
- Then, for any field R, $H_1(M, R)$ is an $R[t, t^{-1}]$-module of the following form:

$$\oplus \frac{R[t, t^{-1}]}{(\lambda_i)}.$$

Plane curves and Alexander type invariants VIII

- In the above terms, the global Alexander polynomial has the following definition:

Definition-global Alexander polynomial
$\Delta_C(t) = \Pi \lambda_i.$

- One could remark that it is defined only up to units in $R[t, t^{-1}]$, so we can normalize it so that it is monic, has non-zero constant term and only positive powers.

- As other computational tools, we mention the usual Fox calculus which use the presentation of π_1 from the Zariski-van Kampen theorem,

- and Libgober description of the global Alexander polynomial, in terms of the local contributions of the singularities. However, we must emphasize that these local contributions also takes into account information on the relative position of the singularities.

Plane curves and Alexander type invariants VIII

- In the above terms, the global Alexander polynomial has the following definition:

> Definition-global Alexander polynomial
> $\Delta_C(t) = \Pi \lambda_i$.

- One could remark that it is defined only up to units in $R[t, t^{-1}]$, so we can normalize it so that it is monic, has non-zero constant term and only positive powers.

- As other computational tools, we mention the usual Fox calculus which use the presentation of π_1 from the Zariski-van Kampen theorem,

- and Libgober description of the global Alexander polynomial, in terms of the local contributions of the singularities. However, we must emphasize that these local contributions also takes into account information on the relative position of the singularities.

Plane curves and Alexander type invariants VIII

- In the above terms, the global Alexander polynomial has the following definition:

Definition-global Alexander polynomial
$\Delta_C(t) = \Pi \lambda_i$.

- One could remark that it is defined only up to units in $R[t, t^{-1}]$, so we can normalize it so that it is monic, has non-zero constant term and only positive powers.
- As other computational tools, we mention the usual Fox calculus which use the presentation of π_1 from the Zariski-van Kampen theorem,
- and Libgober description of the global Alexander polynomial, in terms of the local contributions of the singularities. However, we must emphasize that these local contributions also takes into account information on the relative position of the singularities.

Plane curves and Alexander type invariants VIII

- In the above terms, the global Alexander polynomial has the following definition:

Definition-global Alexander polynomial

$\Delta_C(t) = \Pi \lambda_i$.

- One could remark that it is defined only up to units in $R[t, t^{-1}]$, so we can normalize it so that it is monic, has non-zero constant term and only positive powers.
- As other computational tools, we mention the usual Fox calculus which use the presentation of π_1 from the Zariski-van Kampen theorem,
- and Libgober description of the global Alexander polynomial, in terms of the local contributions of the singularities. However, we must emphasize that these local contributions also takes into account information on the relative position of the singularities.

Plane curves and Alexander type invariants IX

- For example in the case of Zariski's sextics, the global Alexander polynomials are 1 or $t^2 - t + 1$, depending if the cusps are or not on a conic.

- As a last remark, the global Alexander polynomial depends only on the fundamental group of the complement. However, it is known that the braid monodromy determines the full homotopy type of the complement [L]. The next section introduces the idea of Libgober which use the finer braid monodromy to construct invariants that sees more deeply the structure of the complement than the fundamental group.

Plane curves and Alexander type invariants IX

- For example in the case of Zariski's sextics, the global Alexander polynomials are 1 or $t^2 - t + 1$, depending if the cusps are or not on a conic.

- As a last remark, the global Alexander polynomial depends only on the fundamental group of the complement. However, it is known that the braid monodromy determines the full homotopy type of the complement [L]. The next section introduces the idea of Libgober which use the finer braid monodromy to construct invariants that sees more deeply the structure of the complement than the fundamental group.

The Libgober idea I

- The idea of Libgober is very simple and is inspired from the usual description of the Alexander polynomial as the greatest common divisor of the minors of a presentation matrix for the Alexander module over $R[t, t^{-1}]$.

- Recall the braid monodromy morphism $\theta : F_r \to B_d$ where r is the number of critical values of a generic projection and d is the degree of the curve.

- Take as a second input an n dimensional representation ρ of the braid group B_d over the ring $R[t, t^{-1}]$.

- For α_i with $i = 1...r$ a generating system of $F_r \simeq \pi_1(L \setminus crit. val.)$, take the $(r \cdot n, n)$ matrix

$$A := \bigoplus (\rho(\theta(\alpha_i)) - Id)$$

over $R[t, t^{-1}]$.

THE LIBGOBER IDEA I

- The idea of Libgober is very simple and is inspired from the usual description of the Alexander polynomial as the greatest common divisor of the minors of a presentation matrix for the Alexander module over $R[t, t^{-1}]$.

- Recall the braid monodromy morphism $\theta : F_r \to B_d$ where r is the number of critical values of a generic projection and d is the degree of the curve.

- Take as a second input an n dimensional representation ρ of the braid group B_d over the ring $R[t, t^{-1}]$.

- For α_i with $i = 1...r$ a generating system of $F_r \simeq \pi_1(L \setminus \text{crit. val.})$, take the $(r \cdot n, n)$ matrix

$$A := \bigoplus (\rho(\theta(\alpha_i)) - Id)$$

over $R[t, t^{-1}]$.

THE LIBGOBER IDEA I

- The idea of Libgober is very simple and is inspired from the usual description of the Alexander polynomial as the greatest common divisor of the minors of a presentation matrix for the Alexander module over $R[t, t^{-1}]$.

- Recall the braid monodromy morphism $\theta : F_r \to B_d$ where r is the number of critical values of a generic projection and d is the degree of the curve.

- Take as a second input an n dimensional representation ρ of the braid group B_d over the ring $R[t, t^{-1}]$.

- For α_i with $i = 1...r$ a generating system of $F_r \simeq \pi_1(L \setminus crit. val.)$, take the $(r \cdot n, n)$ matrix

$$A := \bigoplus (\rho(\theta(\alpha_i)) - Id)$$

over $R[t, t^{-1}]$.

The Libgober idea I

- The idea of Libgober is very simple and is inspired from the usual description of the Alexander polynomial as the greatest common divisor of the minors of a presentation matrix for the Alexander module over $R[t, t^{-1}]$.

- Recall the braid monodromy morphism $\theta : F_r \to B_d$ where r is the number of critical values of a generic projection and d is the degree of the curve.

- Take as a second input an n dimensional representation ρ of the braid group B_d over the ring $R[t, t^{-1}]$.

- For α_i with $i = 1...r$ a generating system of $F_r \simeq \pi_1(L \setminus crit. val.)$, take the $(r \cdot n, n)$ matrix

$$A := \bigoplus (\rho(\theta(\alpha_i)) - Id)$$

over $R[t, t^{-1}]$.

The Libgober idea II

- With the above notations, the Libgober invariant $P(C, \rho)$ is defined as follows:

Definition

$P(C, \rho)$ is the greatest common divisor of minors of order n in the matrix A.

- Using a naturality lemma for curves in the same connected component of an equi-singular family of plane curves and two generic projections for them, Libgober proves the following:

Theorem

$P(C, \rho)$ does not depend on the choices made for the braid monodromy and is constant on connected components of an equi-singular family.

- Remind that the choices were L and $P_0 \in L_\infty$.

The Libgober idea II

- With the above notations, the Libgober invariant $P(C, \rho)$ is defined as follows:

Definition

$P(C, \rho)$ is the greatest common divisor of minors of order n in the matrix A.

- Using a naturality lemma for curves in the same connected component of an equi-singular family of plane curves and two generic projections for them, Libgober proves the following:

Theorem

$P(C, \rho)$ does not depend on the choices made for the braid monodromy and is constant on connected components of an equi-singular family.

- Remind that the choices were L and $P_0 \in L_\infty$.

The Libgober idea II

- With the above notations, the Libgober invariant $P(C, \rho)$ is defined as follows:

Definition

$P(C, \rho)$ is the greatest common divisor of minors of order n in the matrix A.

- Using a naturality lemma for curves in the same connected component of an equi-singular family of plane curves and two generic projections for them, Libgober proves the following:

Theorem

$P(C, \rho)$ does not depend on the choices made for the braid monodromy and is constant on connected components of an equi-singular family.

- Remind that the choices were L and $P_0 \in L_\infty$.

The Libgober idea III

- From the above viewpoint, the Alexander polynomial enter into this picture trough the reduced Burau representation:

$$\bar{\rho} : B_d \to Aut(R[t, t^{-1}]^{d-1}).$$

- In this context, Libgober proved the following:

THEOREM

$P(C, \bar{\rho})(t) = \Delta_C(t) \cdot (1 + t + ... + t^{d-1}).$

The Libgober idea III

- From the above viewpoint, the Alexander polynomial enter into this picture trough the reduced Burau representation:

$$\bar{\rho} : B_d \to Aut(R[t, t^{-1}]^{d-1}).$$

- In this context, Libgober proved the following:

Theorem

$P(C, \bar{\rho})(t) = \Delta_C(t) \cdot (1 + t + ... + t^{d-1}).$

Another example: the Krammer polynomial I

- Along the same lines, in 2017, Aktas-Cellat-Gurdogan considered another famaous braid group representation, namely the Krammer's one.

- The Krammer representation $K_n(t, q)$ of the braid group B_n is an $\binom{n}{2}$-dimensional linear representation over the ring of Laurent polynomials in 2 variables $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$. Note that $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ is an UFD.

- Its main importance is due to the fact that it is faithful, giving therefore a positive answer to the question on the linearity of the braid groups.

- In their paper [ACG], the Krammer representation is putted into the Libgober recipe, providing a 2-variable polynomial associated to plane curves.

ANOTHER EXAMPLE: THE KRAMMER POLYNOMIAL I

- Along the same lines, in 2017, Aktas-Cellat-Gurdogan considered another famaous braid group representation, namely the Krammer's one.

- The Krammer representation $K_n(t,q)$ of the braid group B_n is an $\binom{n}{2}$-dimensional linear representation over the ring of Laurent polynomials in 2 variables $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$. Note that $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ is an UFD.

- Its main importance is due to the fact that it is faithful, giving therefore a positive answer to the question on the linearity of the braid groups.

- In their paper [ACG], the Krammer representation is putted into the Libgober recipe, providing a 2-variable polynomial associated to plane curves.

Another example: the Krammer polynomial I

- Along the same lines, in 2017, Aktas-Cellat-Gurdogan considered another famaous braid group representation, namely the Krammer's one.

- The Krammer representation $K_n(t, q)$ of the braid group B_n is an $\binom{n}{2}$-dimensional linear representation over the ring of Laurent polynomials in 2 variables $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$. Note that $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ is an UFD.

- Its main importance is due to the fact that it is faithful, giving therefore a positive answer to the question on the linearity of the braid groups.

- In their paper [ACG], the Krammer representation is putted into the Libgober recipe, providing a 2-variable polynomial associated to plane curves.

Another example: the Krammer polynomial I

- Along the same lines, in 2017, Aktas-Cellat-Gurdogan considered another famaous braid group representation, namely the Krammer's one.

- The Krammer representation $K_n(t, q)$ of the braid group B_n is an $\binom{n}{2}$-dimensional linear representation over the ring of Laurent polynomials in 2 variables $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$. Note that $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ is an UFD.

- Its main importance is due to the fact that it is faithful, giving therefore a positive answer to the question on the linearity of the braid groups.

- In their paper [ACG], the Krammer representation is putted into the Libgober recipe, providing a 2-variable polynomial associated to plane curves.

Another example: the Krammer polynomial II

Definition of the Krammer polynomial

The Krammer polynomial associated to the plane curve C of degree d is

$$k_C(t, q) := P(C, K_d)(t, q).$$

- Their main application concerns the n-gonal curves:

Definition

An n-gonal curve C in $\mathbb{C}^2$ is one whose equation $F(x, y) = 0$ has the property that $deg_y(F) = n$.

- An n-gonal curve is completely reducible if F splits as a product of n linear factors in y.
- In these terms, the main result in [ACG] is the following:

Another example: the Krammer polynomial II

Definition of the Krammer polynomial

The Krammer polynomial associated to the plane curve C of degree d is

$$k_C(t, q) := P(C, K_d)(t, q).$$

- Their main application concerns the *n*-gonal curves:

Definition

An *n*-gonal curve C in $\mathbb{C}^2$ is one whose equation $F(x, y) = 0$ has the property that $deg_y(F) = n$.

- An *n*-gonal curve is completely reducible if F splits as a product of n linear factors in y.
- In these terms, the main result in [ACG] is the following:

Another example: the Krammer polynomial II

Definition of the Krammer polynomial

The Krammer polynomial associated to the plane curve C of degree d is

$$k_C(t, q) := P(C, K_d)(t, q).$$

- Their main application concerns the *n*-gonal curves:

Definition

An *n*-gonal curve C in $\mathbb{C}^2$ is one whose equation $F(x, y) = 0$ has the property that $deg_y(F) = n$.

- An *n*-gonal curve is completely reducible if F splits as a product of *n* linear factors in y.
- In these terms, the main result in [ACG] is the following:

Another example: the Krammer polynomial III

Theorem

The Krammer polynomial for an *n*-gonal completely reducible curve *C* with exactly 1 singular fiber is:

$$k_C(t, q) = (t^{2d} q^{6d} - 1)^{\binom{n}{2}}.$$

- The meaning of the supplementary condition concerning the singular fiber is the following: a fiber of the *x*-projection is singular if it intersects the union of *C* with the section at infinity in less than $n + 1$ points.

THE REPRESENTATION THEORY OF THE sl_2 QUANTUM GROUP I

- In this part we shall describe the representation of the braid group over a Laurent ring with 2 variables, which comes from the world of quantum groups. In fact, these representations are closely connected with the construction of the coloured Jones polynomials in knot theory.

- Let q, s parameters and consider the ring

$$\mathbb{L}_s := \mathbb{Z}[q^{\pm 1}, s^{\pm 1}].$$

- The sl_2 quantum group we shall work with is defined as follows: $U_q(sl(2))$ is the algebra over $\mathbb{L}_s$ generated by the elements $\{E, F^{(n)}, K^{\pm 1} \mid n \in \mathbb{N}^*\}$ with the folowing relations:

$$KK^{-1} = K^{-1}K = 1; \quad KE = q^2 EK; \quad KF^{(n)} = q^{-2n}F^{(n)}K$$

$$F^{(n)}F^{(m)} = \begin{bmatrix} n + m \\ n \end{bmatrix}_q F^{(n+m)}.$$

THE REPRESENTATION THEORY OF THE sl_2 QUANTUM GROUP I

- In this part we shall describe the representation of the braid group over a Laurent ring with 2 variables, which comes from the world of quantum groups. In fact, these representations are closely connected with the construction of the coloured Jones polynomials in knot theory.

- Let q, s parameters and consider the ring

$$\mathbb{L}_s := \mathbb{Z}[q^{\pm 1}, s^{\pm 1}].$$

- The sl_2 quantum group we shall work with is defined as follows: $U_q(sl(2))$ is the algebra over $\mathbb{L}_s$ generated by the elements $\{E, F^{(n)}, K^{\pm 1} \mid n \in \mathbb{N}^*\}$ with the folowing relations:

$$KK^{-1} = K^{-1}K = 1; \quad KE = q^2 EK; \quad KF^{(n)} = q^{-2n}F^{(n)}K$$

$$F^{(n)}F^{(m)} = \begin{bmatrix} n + m \\ n \end{bmatrix}_q F^{(n+m)}.$$

THE REPRESENTATION THEORY OF THE sl_2 QUANTUM GROUP I

- In this part we shall describe the representation of the braid group over a Laurent ring with 2 variables, which comes from the world of quantum groups. In fact, these representations are closely connected with the construction of the coloured Jones polynomials in knot theory.
- Let q, s parameters and consider the ring

$$\mathbb{L}_s := \mathbb{Z}[q^{\pm 1}, s^{\pm 1}].$$

- The sl_2 quantum group we shall work with is defined as follows: $U_q(sl(2))$ is the algebra over $\mathbb{L}_s$ generated by the elements $\{E, F^{(n)}, K^{\pm 1} \mid n \in \mathbb{N}^*\}$ with the folowing relations:

$$KK^{-1} = K^{-1}K = 1; \quad KE = q^2 EK; \quad KF^{(n)} = q^{-2n}F^{(n)}K$$

$$F^{(n)}F^{(m)} = \begin{bmatrix} n+m \\ n \end{bmatrix}_q F^{(n+m)}.$$

THE REPRESENTATION THEORY OF THE sl_2 QUANTUM GROUP II

- For the definition of the quantum bracket we used the following notations:

-

$$\{x\} := q^x - q^{-x} \quad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}$$

$$[n]_q! = [1]_q[2]_q...[n]_q$$

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]_q!}{[n-j]_q![j]_q!}.$$

- Also, lets considers V the Verma module freely generated over $\mathbb{L}_s$ by elements $v_0, v_1, ...$ with the following $U_q(sl(2))$ action:

$$K \cdot v_j = sq^{-2j} v_j$$

$$E \cdot v_j = v_{j-1}$$

$$F^{(n)} \cdot v_j = \begin{bmatrix} n+j \\ j \end{bmatrix}_q \prod_{k=0}^{n-1} (sq^{-k-j} - s^{-1}q^{k+j}) v_{j+n}.$$

THE REPRESENTATION THEORY OF THE sl_2 QUANTUM GROUP II

- For the definition of the quantum bracket we used the following notations:

-
$$\{x\} := q^x - q^{-x} \qquad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}$$
$$[n]_q! = [1]_q[2]_q...[n]_q$$
$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]_q!}{[n-j]_q![j]_q!}.$$

- Also, lets considers V the Verma module freely generated over $\mathbb{L}_s$ by elements $v_0, v_1, ...$ with the following $U_q(sl(2))$ action:

$$K \cdot v_j = sq^{-2j}v_j$$

$$E \cdot v_j = v_{j-1}$$

$$F^{(n)} \cdot v_j = \begin{bmatrix} n+j \\ j \end{bmatrix}_q \prod_{k=0}^{n-1}(sq^{-k-j} - s^{-1}q^{k+j})v_{j+n}.$$

THE REPRESENTATION THEORY OF THE sl_2 QUANTUM GROUP II

- For the definition of the quantum bracket we used the following notations:
-

$$\{x\} := q^x - q^{-x} \quad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}$$

$$[n]_q! = [1]_q[2]_q...[n]_q$$

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]_q!}{[n-j]_q![j]_q!}.$$

- Also, lets considers V the Verma module freely generated over $\mathbb{L}_s$ by elements $v_0, v_1, ...$ with the following $U_q(sl(2))$ action:

$$K \cdot v_j = sq^{-2j}v_j$$

$$E \cdot v_j = v_{j-1}$$

$$F^{(n)} \cdot v_j = \begin{bmatrix} n+j \\ j \end{bmatrix}_q \prod_{k=0}^{n-1}(sq^{-k-j} - s^{-1}q^{k+j})v_{j+n}.$$

The representation theory of the sl_2 quantum group III

- Using the R-matrix of $U_q(sl(2))$, a standard construction proves:

Theorem

The braid group B_n acts on $V^{\otimes n}$ as a module over $\mathbb{L}_s$. Moreover, this action commutes with the $U_q(sl(2))$-action.

- The last assertion above implies that the braid group B_n acts also on the usual weight and highest weight spaces:

$$V_{n,m} = ker(K - s^n q^{-2l} Id) \subset V^{\otimes n}$$

$$W_{n,m} = ker(E) \cap ker(K - s^n q^{-2l} Id) \subset V^{\otimes n}.$$

- Due to the fact that for $m = n(N - 1)$ (where N is the label of V_N, the finite N-dimensional $U_q(sl(2))$-representation) the B_n-action on $W_{n,m}$ is connected with the coloured Jones polynomial, we consider the following:

THE REPRESENTATION THEORY OF THE sl_2 QUANTUM GROUP III

- Using the R-matrix of $U_q(sl(2))$, a standard construction proves:

THEOREM

The braid group B_n acts on $V^{\otimes n}$ as a module over $\mathbb{L}_s$. Moreover, this action commutes with the $U_q(sl(2))$-action.

- The last assertion above implies that the braid group B_n acts also on the usual weight and highest weight spaces:

$$V_{n,m} = ker(K - s^n q^{-2l} Id) \subset V^{\otimes n}$$

$$W_{n,m} = ker(E) \cap ker(K - s^n q^{-2l} Id) \subset V^{\otimes n}.$$

- Due to the fact that for $m = n(N-1)$ (where N is the label of V_N, the finite N-dimensional $U_q(sl(2))$-representation) the B_n-action on $W_{n,m}$ is connected with the coloured Jones polynomial, we consider the following:

THE REPRESENTATION THEORY OF THE sl_2 QUANTUM GROUP III

- Using the R-matrix of $U_q(sl(2))$, a standard construction proves:

THEOREM

The braid group B_n acts on $V^{\otimes n}$ as a module over $\mathbb{L}_s$. Moreover, this action commutes with the $U_q(sl(2))$-action.

- The last assertion above implies that the braid group B_n acts also on the usual weight and highest weight spaces:

$$V_{n,m} = ker(K - s^n q^{-2l} Id) \subset V^{\otimes n}$$

$$W_{n,m} = ker(E) \cap ker(K - s^n q^{-2l} Id) \subset V^{\otimes n}.$$

- Due to the fact that for $m = n(N-1)$ (where N is the label of V_N, the finite N-dimensional $U_q(sl(2))$-representation) the B_n-action on $W_{n,m}$ is connected with the coloured Jones polynomial, we consider the following:

THE REPRESENTATION THEORY OF THE sl_2 QUANTUM GROUP IV

DEFINITION

The braid group B_n action on $W_{n,m}$ as a module over $\mathbb{L}_s$ is the coloured Jones action denoted by ρ_{cJ}.

THE LAWRENCE REPRESENTATIONS I

- Another B_n-action over the Laurent ring in two variables comes from a totally different direction: the topology of the configuration spaces in the punctured disk and of their coverings.

- Roughly speaking B_n will acts on a certain sub-space $H_{n.m}$ in the homology of a covering. In full generality it was considered by Lawrence and Bigelow. For $m = 2$ it is obtained the Krammer representation which was initially been introduced by purely algebraic tools.

- The construction goes as follows: (it will be a slight overlap with Cristina's talk)

- Let $n, m \in \mathbb{N}$ be two natural numbers. Let us denote by

$$D_n := \mathbb{D}^2 \setminus \{p_1, ..., p_n\}$$

the n-punctured disc, where $\mathbb{D}^2 \subseteq \mathbb{C}$ is the unit closed disk (with boundary) and $\{p_1, ..., p_n\}$ are n distinct points in its interior, which are also on the real axis.

The Lawrence representations I

- Another B_n-action over the Laurent ring in two variables comes from a totally different direction: the topology of the configuration spaces in the punctured disk and of their coverings.

- Roughly speaking B_n will acts on a certain sub-space $H_{n,m}$ in the homology of a covering. In full generality it was considered by Lawrence and Bigelow. For $m = 2$ it is obtained the Krammer representation which was initially been introduced by purely algebraic tools.

- The construction goes as follows: (it will be a slight overlap with Cristina's talk)

- Let $n, m \in \mathbb{N}$ be two natural numbers. Let us denote by

$$D_n := \mathbb{D}^2 \setminus \{p_1, ..., p_n\}$$

the n-punctured disc, where $\mathbb{D}^2 \subseteq \mathbb{C}$ is the unit closed disk (with boundary) and $\{p_1, ..., p_n\}$ are n distinct points in its interior, which are also on the real axis.

THE LAWRENCE REPRESENTATIONS I

- Another B_n-action over the Laurent ring in two variables comes from a totally different direction: the topology of the configuration spaces in the punctured disk and of their coverings.

- Roughly speaking B_n will acts on a certain sub-space $H_{n.m}$ in the homology of a covering. In full generality it was considered by Lawrence and Bigelow. For $m = 2$ it is obtained the Krammer representation which was initially been introduced by purely algebraic tools.

- The construction goes as follows: (it will be a slight overlap with Cristina's talk)

- Let $n, m \in \mathbb{N}$ be two natural numbers. Let us denote by

$$D_n := \mathbb{D}^2 \setminus \{p_1, ..., p_n\}$$

the n-punctured disc, where $\mathbb{D}^2 \subseteq \mathbb{C}$ is the unit closed disk (with boundary) and $\{p_1, ..., p_n\}$ are n distinct points in its interior, which are also on the real axis.

The Lawrence representations I

- Another B_n-action over the Laurent ring in two variables comes from a totally different direction: the topology of the configuration spaces in the punctured disk and of their coverings.

- Roughly speaking B_n will acts on a certain sub-space $H_{n.m}$ in the homology of a covering. In full generality it was considered by Lawrence and Bigelow. For $m = 2$ it is obtained the Krammer representation which was initially been introduced by purely algebraic tools.

- The construction goes as follows: (it will be a slight overlap with Cristina's talk)

- Let $n, m \in \mathbb{N}$ be two natural numbers. Let us denote by

$$D_n := \mathbb{D}^2 \setminus \{p_1, ..., p_n\}$$

the n-punctured disc, where $\mathbb{D}^2 \subseteq \mathbb{C}$ is the unit closed disk (with boundary) and $\{p_1, ..., p_n\}$ are n distinct points in its interior, which are also on the real axis.

THE LAWRENCE REPRESENTATIONS II

- Consider the configuration space of m unordered points in D_n:

$$C_{n,m} = Conf_m(D_n) = (D_n^m \setminus \Delta)/Sym_m$$

where $\Delta = \{x = (x_1, ..., x_n) \in D_n^m | \exists i, j \text{ such that } x_i = x_j\}$. In the sequel we will use the homology of a certain covering space associated to $C_{n,m}$. We will define it using a certain local system as follows.

- Let

$$\rho : \pi_1(C_{n,m}) \to H_1(C_{n,m})$$

be the abelianisation map. Then, for any $n \in \mathbb{N}$ and $m \in \mathbb{N}$, $m \geq 2$ one has:

$$H_1(C_{n,m}) \simeq \mathbb{Z}^n \oplus \mathbb{Z}$$

- Consider the function $\epsilon : \mathbb{Z}^n \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$
$$< x > < d >$$
given by the augmentation of the first n components

$$\epsilon((x_1, ..., x_n), y) = (x_1 + ... + x_n, y).$$

The Lawrence representations II

- Consider the configuration space of m unordered points in D_n:

$$C_{n,m} = Conf_m(D_n) = (D_n^m \setminus \Delta)/Sym_m$$

where $\Delta = \{x = (x_1, ..., x_n) \in D_n^m | \exists\ i, j \text{ such that } x_i = x_j\}$. In the sequel we will use the homology of a certain covering space associated to $C_{n,m}$. We will define it using a certain local system as follows.

- Let

$$\rho : \pi_1(C_{n,m}) \rightarrow H_1(C_{n,m})$$

be the abelianisation map. Then, for any $n \in \mathbb{N}$ and $m \in \mathbb{N}$, $m \geq 2$ one has:

$$H_1(C_{n,m}) \simeq \mathbb{Z}^n \oplus \mathbb{Z}$$

- Consider the function $\epsilon : \mathbb{Z}^n \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$
$$< x > < d >$$
given by the augmentation of the first n components

$$\epsilon((x_1, ..., x_n), y) = (x_1 + ... + x_n, y).$$

The Lawrence representations II

- Consider the configuration space of m unordered points in D_n:

$$C_{n,m} = Conf_m(D_n) = (D_n^m \setminus \Delta)/Sym_m$$

where $\Delta = \{x = (x_1, ..., x_n) \in D_n^m | \exists i, j \text{ such that } x_i = x_j\}$. In the sequel we will use the homology of a certain covering space associated to $C_{n,m}$. We will define it using a certain local system as follows.

- Let

$$\rho : \pi_1(C_{n,m}) \to H_1(C_{n,m})$$

be the abelianisation map. Then, for any $n \in \mathbb{N}$ and $m \in \mathbb{N}$, $m \geq 2$ one has:

$$H_1(C_{n,m}) \simeq \mathbb{Z}^n \oplus \mathbb{Z}$$

- Consider the function $\epsilon : \mathbb{Z}^n \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$
$$<x><d>$$

given by the augmentation of the first n components

$$\epsilon((x_1, ..., x_n), y) = (x_1 + ... + x_n, y).$$

THE LAWRENCE REPRESENTATIONS III

- We consider the following local system $\phi : \pi_1(C_{n,m}) \to \mathbb{Z} \oplus \mathbb{Z}$ given by:

$$< x >< d >$$

$$\phi = \epsilon \circ \rho.$$

Let $\tilde{C}_{n,m}$ be the covering space of $C_{n,m}$ which corresponds to $Ker(\phi)$, and denote by $\pi : \tilde{C}_{n,m} \to C_{n,m}$ the projection map associated to it.

- Now, we want to see that the homology of the covering space of the configuration space in the puncture disc has the feature of carrying a braid group action. We remind that the braid group is the mapping class group of the punctured disc relative to its boundary:

$$B_n = MCG(D_n) = Homeo^+(D_n, \partial)/\text{isotopy}$$

The Lawrence representations III

- We consider the following local system $\phi : \pi_1(C_{n,m}) \to \mathbb{Z} \oplus \mathbb{Z}$ given by:

$$< x >< d >$$

$$\phi = \epsilon \circ \rho.$$

  Let $\tilde{C}_{n,m}$ be the covering space of $C_{n,m}$ which corresponds to $Ker(\phi)$, and denote by $\pi : \tilde{C}_{n,m} \to C_{n,m}$ the projection map associated to it.

- Now, we want to see that the homology of the covering space of the configuration space in the puncture disc has the feature of carrying a braid group action. We remind that the braid group is the mapping class group of the punctured disc relative to its boundary:

$$B_n = MCG(D_n) = Homeo^+(D_n, \partial)/\text{isotopy}$$

The Lawrence representations IV

- Then B_n will act onto the configuration space $C_{n,m}$ by homeomorphisms and it will induce an action on its fundamental group

$$B_n \curvearrowright \pi_1(C_{n,m}).$$

Proposition

This braid group action behaves well with respect to the local system ϕ, and it can be lifted to an action onto the homology of the convering. Moreover, this action is compatible with the action of the deck transformations and one has that:

$$B_n \curvearrowright H_m^{lf}(\tilde{C}_{n,m}, \mathbb{Z}) \text{ (as a module over } \mathbb{Z}[x^{\pm}, d^{\pm}]).$$

- The Lawrence representation will be defined on a certain subspace of the middle dimensional Borel-Moore homology of the covering $\tilde{C}_{n,m}$ described above.

THE LAWRENCE REPRESENTATIONS IV

- Then B_n will act onto the configuration space $C_{n,m}$ by homeomorphisms and it will induce an action on its fundamental group

$$B_n \curvearrowright \pi_1(C_{n,m}).$$

PROPOSITION

This braid group action behaves well with respect to the local system ϕ, and it can be lifted to an action onto the homology of the convering. Moreover, this action is compatible with the action of the deck transformations and one has that:

$$B_n \curvearrowright H_m^{lf}(\tilde{C}_{n,m}, \mathbb{Z}) \text{ (as a module over } \mathbb{Z}[x^{\pm}, d^{\pm}]).$$

- The Lawrence representation will be defined on a certain subspace of the middle dimensional Borel-Moore homology of the covering $\tilde{C}_{n,m}$ described above.

- More precisely, this subspace denoted $H_{n,m}$ is spanned by certain explicitly constructed classes indexed by

$$E_{n,m} = \{e = (e_1, ..., e_{n-1}) \in \mathbb{N}^n \mid e_1 + ... + e_{n-1} = m\}.$$

With these notations we have:

Lawrence representation

The braid group action $B_n \curvearrowright H_m^{\text{lf}}(\tilde{C}_{n,m}, \mathbb{Z})$ preserves the subspace $H_{n,m}$ and there is a braid group representation:

$$l_{n,m} : B_n \to Aut(H_{n,m}, \mathbb{Z}[x^{\pm 1}, d^{\pm 1}])$$

called Lawrence representation.

We mention that the dimension of the Lawrence representation $H_{n,m}$ is $\binom{n+m-2}{2}$ if $m \geq 2$. For $m = 1$, $H_{n,1}$ is the reduced Burau representation.

The coloured Jones and Lawrence polynomials for plane algebraic curves I

- Putting ρ_{cJ} and $l_{n.m}$ into the Libgober algorithm, we obtain the following:

Theorem (-, C. Anhgel, M. Palmer)

There exists two polynomial invariants, the coloured Jones and the Lawrence one, $P(C, \rho_{cJ})(q, s)$ and $P(C, l_{n.m})(x, d)$ which are constant in connected components of equi-singular families of plane curves. After a convenient change of variables, the two coincides.

- The main interest for considering these new invariants is to try to detect by their use non-equivalent plane curves with the same values for other invariants for example π_1 or the Alexander polynomial.

- The next section is devoted to such pathological situations where the classical invariants are useless.

The coloured Jones and Lawrence polynomials for plane algebraic curves I

- Putting ρ_{cJ} and $l_{n.m}$ into the Libgober algorithm, we obtain the following:

Theorem (-, C. Anhgel, M. Palmer)

There exists two polynomial invariants, the coloured Jones and the Lawrence one, $P(C, \rho_{cJ})(q, s)$ and $P(C, l_{n.m})(x, d)$ which are constant in connected components of equi-singular families of plane curves. After a convenient change of variables, the two coincides.

- The main interest for considering these new invariants is to try to detect by their use non-equivalent plane curves with the same values for other invariants for example π_1 or the Alexander polynomial.
- The next section is devoted to such pathological situations where the classical invariants are useless.

The coloured Jones and Lawrence polynomials for plane algebraic curves I

- Putting ρ_{cJ} and $l_{n.m}$ into the Libgober algorithm, we obtain the following:

Theorem (-, C. Anhgel, M. Palmer)

There exists two polynomial invariants, the coloured Jones and the Lawrence one, $P(C, \rho_{cJ})(q, s)$ and $P(C, l_{n.m})(x, d)$ which are constant in connected components of equi-singular families of plane curves. After a convenient change of variables, the two coincides.

- The main interest for considering these new invariants is to try to detect by their use non-equivalent plane curves with the same values for other invariants for example π_1 or the Alexander polynomial.
- The next section is devoted to such pathological situations where the classical invariants are useless.

Zariski pairs and future directions I

- A Zariski pair consists of two curves C_1, C_2 with the same type of singularities, with homeomorphic tubular neighborhoods but non-homeomorphic as embedded curves.

- The classical example seen at the beginning is due to Zariski and consists of two sextics with 6 cusps. The good point for that pair is the fact that both the π_1 and the Alexander polynomial detect them. In particular, the fundamental group of the complement is computable.

- However, there exists many examples of Zariski pairs which are not detected neither by the homeomorphism type of the complement.

- This fact, for example is in striking contrast to the knot theory situation, where the Gordon-Luck theorem implies that the homeo-type of the complement determines the knot type.

Zariski pairs and future directions I

- A Zariski pair consists of two curves C_1, C_2 with the same type of singularities, with homeomorphic tubular neighborhoods but non-homeomorphic as embedded curves.

- The classical example seen at the beginning is due to Zariski and consists of two sextics with 6 cusps. The good point for that pair is the fact that both the π_1 and the Alexander polynomial detect them. In particular, the fundamental group of the complement is computable.

- However, there exists many examples of Zariski pairs which are not detected neither by the homeomorphism type of the complement.

- This fact, for example is in striking contrast to the knot theory situation, where the Gordon-Luck theorem implies that the homeo-type of the complement determines the knot type.

Zariski pairs and future directions I

- A Zariski pair consists of two curves C_1, C_2 with the same type of singularities, with homeomorphic tubular neighborhoods but non-homeomorphic as embedded curves.

- The classical example seen at the beginning is due to Zariski and consists of two sextics with 6 cusps. The good point for that pair is the fact that both the π_1 and the Alexander polynomial detect them. In particular, the fundamental group of the complement is computable.

- However, there exists many examples of Zariski pairs which are not detected neither by the homeomorphism type of the complement.

- This fact, for example is in striking contrast to the knot theory situation, where the Gordon-Luck theorem implies that the homeo-type of the complement determines the knot type.

Zariski pairs and future directions I

- A Zariski pair consists of two curves C_1, C_2 with the same type of singularities, with homeomorphic tubular neighborhoods but non-homeomorphic as embedded curves.

- The classical example seen at the beginning is due to Zariski and consists of two sextics with 6 cusps. The good point for that pair is the fact that both the π_1 and the Alexander polynomial detect them. In particular, the fundamental group of the complement is computable.

- However, there exists many examples of Zariski pairs which are not detected neither by the homeomorphism type of the complement.

- This fact, for example is in striking contrast to the knot theory situation, where the Gordon-Luck theorem implies that the homeo-type of the complement determines the knot type.

Zariski pairs and future directions II

- With the above facts in mind, a natural question could be the following:

Question 1

To find Zariski pairs with the same Alexander polynomial but with different coloured Jones/Lawrence polynomial.

- Due to the fact that by construction, the cJ/L polynomial depend on the braid monodromy, which is in fact a finer invariant than the π_1 itself, a more ambitious question could be:

Question 2

To find Zariski pairs with the same π_1 but with different coloured Jones/Lawrence polynomial.

ZARISKI PAIRS AND FUTURE DIRECTIONS II

- With the above facts in mind, a natural question could be the following:

QUESTION 1

To find Zariski pairs with the same Alexander polynomial but with different coloured Jones/Lawrence polynomial.

- Due to the fact that by construction, the cJ/L polynomial depend on the braid monodromy, which is in fact a finer invariant than the π_1 itself, a more ambitious question could be:

QUESTION 2

To find Zariski pairs with the same π_1 but with different coloured Jones/Lawrence polynomial.

Zariski pairs and future directions III

Last but not least, the world of quantum groups at root of unity offer also a wide class of braid group representation. From this viewpoint one can ask:

Question 3

To study Zariski pairs through the lens of other (non-semisimple) polynomials like the coloured Alexander (ADO).

REFERENCES

- A. Libgober, Invariants of plane algebraic curves via representations of the braid groups, Inventiones mathematicae 95(1),25-30, 1989.
- M. AKTAS, S. CELLAT, H. GURDOGAN, Journal of Singularities Volume 17 (2018), 58-69

THANK YOU!