Special bundles and superstrings

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Abstract

We intend to prezent the class of stable omalous bundles, a construction of them and their applications in superstrings.

Outline

Introduction

- Mirror symmetry is an highly elaborated correspondence, where, for certain pairs of varieties X and Y one are able to understand some classes of invariants of X in terms of invariants of Y of completely different type. One side of this correspondence (let's say on X) is connected with the so-called Gromov-Witten invariants which are encoded in the quantum deformation of the usual cup product in $H^*(X, \mathbb{C})$.
- In recent years, another correspondence of this type -$(0;2)$-mirror symmetry- became an active area of research in connection with the $(0;2)$ nonlinear sigma model from super-strings theory. The main piece in this theory is quantum sheaf cohomology, namely a deformation of the cohomology ring of a sheaf.
- In the first part I intend to explain briefly this subject from the mathematical viewpoint and the importance of the omality condition.
- The second part will be devoted to the construction of stable omalous bundles on surfaces of general type.

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The "classical" quantum cohomology I

- The quantum cohomology is a formal deformation of the cohomology ring $H^*(X, \mathbb{C})$ for a smooth algebraic variety X.
- From the physical viewpoint it seems like if one try to "observe" the variety X not by its points, but by its rational curves.
- The result is a quantum product on $H^*(X, \mathbb{C})[[\mathbf{q}]]$ where $\mathbf{q} = (q_1, q_2, ...)$ is a multi-index whose length is the rank of $H_2(X, \mathbb{Z})$.
- The main ingredient in the construction is Kontsevich's moduli space of stable maps

$$\mathcal{M}(X, \beta)$$

  which parametrize maps $\mathbb{P}^1 \to X$ whose image has class $\beta \in H_2(X, \mathbb{Z})$.
- Also, one should mention that the construction depend on a system of marked points, but we shall ignore this aspect here.

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The "classical" quantum cohomology II

- Having the moduli space of stable maps, the construction of the quantum product on $H^*(X, \mathbb{C})$ goes as follows:
- for two classes ω_1, $\omega_2 \in H^*(X, \mathbb{C})$ the big deal is to define their quantum product

$$\omega_1 \star \omega_2 \in H^*(X, \mathbb{C})[[\mathbf{q}]].$$

- A first observation is that as consequence of Poincare duality, it is enough to define the pairing

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  for any class $\omega_3 \in H^*(X, \mathbb{C})$.
- The pairing can be expanded as a formal sum

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The "classical" quantum cohomology III

- So, we are reduced to the computation of this triple product $< \omega_1\omega_2\omega_3 >_\beta$ for each $\beta \in H_2(X, \mathbb{Z})$.
- This is done by pushing up the three ω_i in the cohomology of $\mathcal{M}(X, \beta)$ and by taking here the cup product of the resulting classes:

$$< \omega_1\omega_2\omega_3 >_\beta = \varphi_\beta^*(\omega_1) \cup \varphi_\beta^*(\omega_2) \cup \varphi_\beta^*(\omega_3),$$

where the push up map $\varphi_\beta : H^*(X, \mathbb{C}) \to H^*(\mathcal{M}(X, \beta), \mathbb{C})$ is constructed using the marked points.
- Even in such a oversimplified picture, we must mention three great difficulties in the full story:
- First of all, the moduli space $\mathcal{M}(X, \beta)$ is not compact at the beginning and its compactification was obtained by Kontsevich. Secondly, this compactification is not a variety but a stack.
- Thirdly, on the compactification one needs a so called virtual fundamental class which is used to give a rigorous meaning for the cup product on $\mathcal{M}(X, \beta)$.

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The "classical" quantum cohomology: examples

- All these difficulties were resolved for large classes of varieties. Below are two simple examples:
- $$QH^*(\mathbb{P}^n) = \frac{\mathbb{C}[x][[q]]}{(x^{n+1} - q)}$$
- $$QH^*(\mathbb{P}^n \times \mathbb{P}^m) = \frac{\mathbb{C}[x, y][[p, q]]}{(x^{n+1} - p, y^{m+1} - q)}.$$

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Quantum cohomology for sheaves I

- Quantum cohomology for sheaves was introduced by Donagi et al. in 2011 [arxiv 1110.3751] in connection with the $(0,2)$ nonlinear sigma model.
- Its construction is similar with that for varieties but has a sheaf E on the variety X as supplementary input.
- The sheaf E has to satisfy certain constraint - the omality condition -

$$c_1(T_X) = c_1(E) \; c_2(T_X) = c_2(E)$$

which imply the vanishing of the Green-Schwarz anomaly.
- An important point is that the omality is a necessary but not sufficient condition for the existence of a quantum sheaf cohomology for E.
- As definition, the quantum sheaf cohomology for E is the structure of a quantum product on

$$QH^*(X,E) := H^*(X, \Lambda^*(E^v)) \otimes \mathbb{C}[[\mathbf{q}]].$$

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- It is a deformation of the usual product on the cohomology of E.
- Remark: For $E = T_X$ the quantum sheaf cohomology of E is the "classical" quantum cohomology of X as one can guess from the Hodge decomposition.
- The construction goes along the same lines as in the "classical" case: one starts with two elements $\omega_1, \omega_2 \in H^*(X, \Lambda^*(E^\vee))$ and we want to define their quantum product

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- The next step is to push up the ω_i $'s$ from $H^*(X, \Lambda^*(E^v))$ to $H^*(\mathcal{M}(X,\beta), \Lambda^*(F^v))$, where F is a certain sheaf over $\mathcal{M}(X,\beta)$ obtained from E.
- For example, in an ideal situation when the moduli space $\mathcal{M}(X,\beta)$ were fine with classifying map

$$\varphi : \mathcal{M}(X,\beta) \times \mathbb{P}^1 \to X,$$

then F would be $R^0\pi_{1*}\varphi*(E)$.
- Anyway, even in the real world where $\mathcal{M}(X,\beta)$ is not fine, such an F exists and the main point is that
- the omality of E imply $\Lambda^{top}(F^v) \simeq K_{\mathcal{M}(X,\beta)}$
- As consequence, if one starts with $\omega_i \in H^{p_i}(X, \Lambda^{q_i}(E^v))$ with $\Sigma p_i = dim\ \mathcal{M}(X,\beta)$ and $\Sigma q_i = rank(F)$ then by pushing up the ω_i $'s$ and taking cup product we arrive in $H^{top}(\mathcal{M}(X,\beta), K) \simeq \mathbb{C}$, producing therefore the desired number $< \omega_1\omega_2\omega_3 >_\beta$.

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Quantum cohomology for sheaves III

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The quadric surface I

- As one could remark, apart the difficulties connected with the "classical" quantum cohomology, at least two new problems can be seen: the construction of the bundle F and the starting point, namely the construction of the omalous bundle E
- In fact, the construction of a quantum sheaf cohomology is known in very few cases.
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- As the starting point on the quadric, the above mentioned authors considers de bundle E as cokernel in the following sequence:

$$0 \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \to E \to 0,$$

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- where A, B, C, D are 2×2 complex matrices and $x = (x_1, x_2)$, $x' = (x'_1, x'_2)$ are homogenous coordinates on the two projective lines.
- Note that the bundle E is a deformation of T_X which correspond to the special values $A = D = I_2$ and $B = C = 0$.
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Motivation for the second part I

- As one can see from the previous part, the first ingredient for the construction of a quantum sheaf cohomology is an omalous bundle.
- However, despite their importance, only few examples are known in the literature. A first class of examples consists of deformations of the tangent bundle T_X which are omalous by trivial reasons.
- On the other hand, as showed by Andreas and Garcia-Fernandez in 2010, the stability of an omalous bundle is a very important property, because such a bundle provide a solution of the so-called Strominger system in super-string theory.
- In this direction, a first systematic attempt to construct stable omalous bundles was done in 2011 by Henni and Jardim in arxiv:1105.5588. They uses monads to construct:
  - stable omalous rank 3 bundles on 3-folds in $\mathbb{P}^4$,
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- Also, in arxiv:1506.01479, Aprodu and Marchitan studied omalous rank 2-bundles on Hirzebruch surfaces.
- The rest of my talk will be devoted to the construction of stable omalous bundles on surfaces, with special emphasis for the case where X is a surface of general type.
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General construction of stable bundles on surfaces I

- In what follows X is a smooth projective surface and L a very ample polarization. The L-stability of a sheaf E means that it has the greatest fraction $\frac{c_1 \cdot L}{rank}$ among all its sub-sheaves.
- The main problem for the moment is the following: fixing the rank r and the first Chern class c_1, to find a computable bound α depending only on r, L and c_1 such that for any $c_2 \geq \alpha$ there is an L-stable vector bundle of rank r with the given Chern classes c_1 and c_2.
- The main result of Li and Qin asserts that for α one can take the following value:
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- The main point in the proof of Li-Qin theorem is the following generalization of the usual Cayley-Bacharach property:

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Consider $r-1$ line bundles $L_1, ..., L_{r-1}$ and 0-cycles $Z_1, ..., Z_{r-1}$ on X; let $W = \oplus(\mathcal{O}_X(L_i) \otimes \mathcal{I}_{Z_i})$. Then, there is a locally free extension in $Ext^1(W, \mathcal{O}_X(L'))$ iff for any $i = 1...(r-1)$, Z_i satisfies the Cayley-Bacharach property with respect to the linear system $\mathcal{O}_X(L_i - L' + K_S)$.

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- As a conclusion with respect to the Li-Qin construction we can state the following:
- Remark: The value of α grows up with the Chern number c_1^2 due to the presence of the h^0-term and to Riemann-Roch.
- Therefore, if their construction can produce omalous bundles, it is better to try on surfaces that satisfy at least

$$c_2 >> c_1^2.$$

- The above inequality, combined with Bogomolov-Miyaoka-Yau inequality $c_1^2 \leq 3c_2$ for surfaces of general type, suggests to look at certain convenient such surfaces.

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Stable omalous bundles on "good" surfaces of general type I

- Viewing the above considerations, we can introduce the following:

Definition

A surface of general type is "good of type (r, L)" if for $c_1 = \pm K_X$ and $c_2 = c_2(X)$, there exists $r \in \mathbb{N}$ and a very ample line bundle L, such that

$$c_2 \geq \alpha(r, c_1, L),$$

where α is the Li-Qin constant introduced before.

- In terms of the above definition, the Li-Qin existence Theorem has the following obvious consequence:

Corollary

On a general type surface X "good of type (r, L)", there exists L-stable omalous vector bundles of rank r.

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- Viewing the above considerations, we can introduce the following:

Definition

A surface of general type is "good of type (r, L)" if for $c_1 = \pm K_X$ and $c_2 = c_2(X)$, there exists $r \in \mathbb{N}$ and a very ample line bundle L, such that

$$c_2 \geq \alpha(r, c_1, L),$$

where α is the Li-Qin constant introduced before.

- In terms of the above definition, the Li-Qin existence Theorem has the following obvious consequence:

Corollary

On a general type surface X "good of type (r, L)", there exists L-stable omalous vector bundles of rank r.

Examples of "good" surfaces I

- This last section is devoted to the illustration of the above results on a concrete class of examples: X_d a smooth surface of degree d in $\mathbb{P}^3$.
- Well known computations gives for X_d the following values of invariants:

$$c_2 = d^3 - 4d^2 + 6d \qquad c_1^2 = d(d-4)^2$$

- Also, by Noether formula,

$$p_g = \chi(\mathcal{O}) - 1 = \frac{c_1^2 + c_2}{12} = \frac{d^3}{6} + \ldots.$$

- So, the leading therm in d which appear in the Li-Qin constant α is

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Examples of "good" surfaces II

- As consequence, for $2 \leq r \leq 6$ and $d >> 0$ we have $c_2 \geq \alpha$.
- So, on can apply the Li-Qin construction, obtaining the following:

Theorem (-,2015)

There is an explicitly computable constant d_0, such that for all $d \geq d_0$ and all $2 \leq r \leq 6$, on any smooth surface $X_d \subset \mathbb{P}^3$ of degree d there exists a stable omalous bundle of rank r.

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Further directions

- As further directions, an open question asked by Donagi et al. in the paper "$(0,2)$ Quantum Cohomology" published in Proceedings of Symposia in Pure Mathematics, Vol 85, 2012, concern the computation of the quantum sheaf cohomology for other sheaves than deformations of the tangent bundle.
- Of course, the above constructed stable omalous bundles are good candidates for such a computation.
- Moreover, due to the range of their ranks, less than 7, this question is very interesting viewing the following:

Conjecture: Guffin, 2011

For omalous bundles E of rank $r \leq 7$ on a smooth variety, the quantum sheaf cohomology $QH^*(X, E)$ exists.

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THANK YOU!