

# QUANTUM SHEAF COHOMOLOGY ON SURFACES OF GENERAL TYPE

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- Quantum sheaf cohomology is a deformation of the cohomology ring of a sheaf.
- In recent years, this subject was an active area of research in connection with the  $(0; 2)$  nonlinear sigma model from super-strings theory.
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- The second part will be devoted to the construction of stable omalous bundles on surfaces of general type.

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- 1 QUANTUM SHEAF COHOMOLOGY
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# THE "CLASSICAL" QUANTUM COHOMOLOGY I

- The quantum cohomology is a formal deformation of the cohomology ring  $H^*(X, \mathbb{C})$  for a smooth algebraic variety  $X$ .
- From the physical viewpoint it appears if one try to "observe" the variety  $X$  not by its points, but by its rational curves.
- The result is a quantum product on  $H^*(X, \mathbb{C})[[\mathbf{q}]]$  where  $\mathbf{q} = (q_1, q_2, \dots)$  is a multi-index whose length is the rank of  $H^2(X, \mathbb{Z})$ .
- The main ingredient in the construction is Kontsevich's moduli space of stable maps

$$\mathcal{M}(X, \beta)$$

which parametrize maps  $\mathbb{P}^1 \rightarrow X$  whose image has class  $\beta \in H^2(X, \mathbb{Z})$ .

- Also, one should mention that the construction depend on a system of marked points, but we shall ignore this aspect here.

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- Having the moduli space of stable maps, the construction of the quantum product on  $H^*(X, \mathbb{C})$  goes as follows:
- for two classes  $\omega_1, \omega_2 \in H^*(X, \mathbb{C})$  the big deal is to define their quantum product

$$\omega_1 \star \omega_2 \in H^*(X, \mathbb{C})[[q]].$$

- A first observation is that as consequence of Poincare duality, it is enough to define the pairing

$$\langle \omega_1 \star \omega_2, \omega_3 \rangle$$

for any class  $\omega_3 \in H^*(X, \mathbb{C})$ .

- The pairing can be expanded as a formal sum

$$\sum_{\beta} \langle \omega_1 \omega_2 \omega_3 \rangle_{\beta} q^{\beta}$$

- where  $\beta$  varies in the free part of  $H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^r$ .

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# THE "CLASSICAL" QUANTUM COHOMOLOGY III

- So, we are reduced to the computation of this triple product  $\langle \omega_1 \omega_2 \omega_3 \rangle_\beta$  for each  $\beta \in H^2(X, \mathbb{Z})$ .
- This is done by pushing up the three  $\omega_i$  in the cohomology of  $\mathcal{M}(X, \beta)$  and by taking here the cup product of the resulting classes:

$$\langle \omega_1 \omega_2 \omega_3 \rangle_\beta = \varphi_\beta^*(\omega_1) \cup \varphi_\beta^*(\omega_2) \cup \varphi_\beta^*(\omega_3),$$

where the push up map  $\varphi_\beta : H^*(X, \mathbb{C}) \rightarrow H^*(\mathcal{M}(X, \beta), \mathbb{C})$  is constructed using the marked points.

- Even in such a oversimplified picture, we must mention three great difficulties in the full story:
- First of all, the moduli space  $\mathcal{M}(X, \beta)$  is not compact at the beginning and its compactification was obtained by Kontsevich. Secondly, this compactification is not a variety but a stack.
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# THE "CLASSICAL" QUANTUM COHOMOLOGY: EXAMPLES

- All these difficulties were resolved for large classes of varieties. Below are two simple examples:

- $$QH^*(\mathbb{P}^n) = \frac{\mathbb{C}[x][[q]]}{(x^{n+1} - q)}$$

- $$QH^*(\mathbb{P}^n \times \mathbb{P}^m) = \frac{\mathbb{C}[x, y][[p, q]]}{(x^{n+1} - p, y^{m+1} - q)}.$$

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# QUANTUM COHOMOLOGY FOR SHEAVES I

- Quantum cohomology for sheaves was introduced by Donagi et al. in 2011 [arxiv 1110.3751] in connection with the  $(0, 2)$  nonlinear sigma model.
- Its construction is similar with that for varieties but has a sheaf  $E$  on the variety  $X$  as supplementary input.
- The sheaf  $E$  has to satisfy certain constraint - the omality condition -

$$c_1(T_X) = c_1(E) \quad c_2(T_X) = c_2(E)$$

which imply the vanishing of the Green-Schwarz anomaly.

- An important point is that the omality is a necessary but not sufficient condition for the existence of a quantum sheaf cohomology for  $E$ .
- As definition, the quantum sheaf cohomology for  $E$  is the structure of a quantum product on

$$QH^*(X, E) := H^*(X, \Lambda^*(E^\vee)) \otimes \mathbb{C}[[q]].$$

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- **Remark:** For  $E = T_X$  the quantum sheaf cohomology of  $E$  is the "classical" quantum cohomology of  $X$  as one can guess from the Hodge decomposition.
- The construction goes along the same lines as in the "classical" case: one starts with two elements  $\omega_1, \omega_2 \in H^*(X, \Lambda^*(E^\vee))$  and we want to define their quantum product

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# QUANTUM COHOMOLOGY FOR SHEAVES III

- The next step is to push up the  $\omega_i$  's from  $H^*(X, \Lambda^*(E^\vee))$  to  $H^*(\mathcal{M}(X, \beta), \Lambda^*(F^\vee))$ , where  $F$  is a certain sheaf over  $\mathcal{M}(X, \beta)$  obtained from  $E$ .
- For example, in an ideal situation when the moduli space  $\mathcal{M}(X, \beta)$  were fine with classifying map

$$\varphi : \mathcal{M}(X, \beta) \times \mathbb{P}^1 \rightarrow X,$$

then  $F$  would be  $R^0\pi_{1*}\varphi^*(E)$ .

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- As one could remark, apart the difficulties connected with the "classical" quantum cohomology, at least to new problems can be seen: the construction of the bundle  $F$  and the starting point, namely the construction of the omalous bundle  $E$
- In fact, the construction of a quantum sheaf cohomology is known in very few cases.
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- As the starting point on the quadric, the above mentioned authors considers de bundle  $E$  as cokernel in the following sequence:

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- where  $A, B, C, D$  are  $2 \times 2$  complex matrices and  $x = (x_1, x_2)$ ,  $x' = (x'_1, x'_2)$  are homogenous coordinates on the two projective lines.
- Note that the bundle  $E$  is a deformation of  $T_X$  which correspond to the special values  $A = D = I_2$  and  $B = C = 0$ .
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# MOTIVATION FOR THE SECOND PART I

- As one can see from the previous part, the first ingredient for the construction of a quantum sheaf cohomology is an omalous bundle.
- However, despite their importance, only few examples are known in the literature. A first class of examples consists of deformations of the tangent bundle  $T_X$  which are omalous by trivial reasons.
- On the other hand, as showed by [Andreas and Garcia-Fernandez in 2010](#), the stability of an omalous bundle is a very important property, because such a bundle provide a solution of the so-called Strominger system in super-string theory.
- In this direction, a first systematic attempt to construct stable omalous bundles was done in 2011 by Henni and Jardim in [arxiv:1105.5588](#). They uses monads to construct:
  - stable omalous rank 3 bundles on 3-folds in  $\mathbb{P}^4$ ,
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- The rest of my talk will be devoted to the construction of stable omalous bundles on surfaces, with special emphasis for the case where  $X$  is a surface of general type.
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# GENERAL CONSTRUCTION OF STABLE BUNDLES ON SURFACES I

- In what follows  $X$  is a smooth projective surface and  $L$  a very ample polarization. The  $L$ -stability of a sheaf  $E$  means that it has the greatest fraction  $\frac{c_1 \cdot L}{\text{rank}}$  among all its sub-sheaves.
- The main problem for the moment is the following: fixing the rank  $r$  and the first Chern class  $c_1$ , to find a computable bound  $\alpha$  depending only on  $r$ ,  $L$  and  $c_1$  such that for any  $c_2 \geq \alpha$  there is an  $L$ -stable vector bundle of rank  $r$  with the given Chern classes  $c_1$  and  $c_2$ .
- The main result of Li and Qin asserts that for  $\alpha$  one can take the following value:
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- The main problem for the moment is the following: fixing the rank  $r$  and the first Chern class  $c_1$ , to find a computable bound  $\alpha$  depending only on  $r$ ,  $L$  and  $c_1$  such that for any  $c_2 \geq \alpha$  there is an  $L$ -stable vector bundle of rank  $r$  with the given Chern classes  $c_1$  and  $c_2$ .
- The main result of Li and Qin asserts that for  $\alpha$  one can take the following value:
- $$\alpha = (r-1)[1 + \max(p_g, h^0(S, \mathcal{O}_X(rL - c_1 + K_X)))] + 4(r-1)^2 \cdot L^2 \\ + (r-1)c_1 \cdot L - \frac{r(r-1)}{2} \cdot L^2,$$
- where  $K_X$  is the canonical class and  $p_g = h^0(X, K_X)$  the geometric genus of  $X$ .

# GENERAL CONSTRUCTION OF STABLE BUNDLES ON SURFACES II

- The main point in the proof of Li-Qin theorem is the following generalization of the usual Cayley-Bacharach property:

## LI-QIN LEMMA

Consider  $r - 1$  line bundles  $L_1, \dots, L_{r-1}$  and 0-cycles  $Z_1, \dots, Z_{r-1}$  on  $X$ ; let  $W = \oplus (\mathcal{O}_X(L_i) \otimes \mathcal{I}_{Z_i})$ . Then, there is a locally free extension in  $\text{Ext}^1(W, \mathcal{O}_X(L'))$  iff for any  $i = 1 \dots (r - 1)$ ,  $Z_i$  satisfies the Cayley-Bacharach property with respect to the linear system  $\mathcal{O}_X(L_i - L' + K_S)$ .

- After that, the desired bundle  $E$  is constructed as an extension

$$0 \rightarrow \mathcal{O}_X(c_1 + (1 - r)L) \rightarrow E \rightarrow \oplus (\mathcal{O}_X(L) \otimes \mathcal{I}_{Z_i}) \rightarrow 0,$$

for a convenient choice of  $r - 1$  reduced 0-cycles  $Z'_i$ 's that ensures the stability of  $E$ .

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# GENERAL CONSTRUCTION OF STABLE BUNDLES ON SURFACES III

- As a conclusion with respect to the Li-Qin construction we can state the following:
- **Remark:** The value of  $\alpha$  grows up with the Chern number  $c_1^2$  due to the presence of the  $h^0$ -term and to Riemann-Roch.
- Therefore, if their construction can produce omalous bundles, it is better to try on surfaces that satisfy at least

$$c_2 \gg c_1^2.$$

- The above inequality, combined with Bogomolov-Miyaoka-Yau inequality  $c_1^2 \leq 3c_2$  for surfaces of general type, suggests to look at certain convenient such surfaces.

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# STABLE OMALOUS BUNDLES ON "GOOD" SURFACES OF GENERAL TYPE I

- Viewing the above considerations, we can introduce the following:

## DEFINITION

A surface of general type is "good of type  $(r, L)$ " if for  $c_1 = \pm K_X$  and  $c_2 = c_2(X)$ , there exists  $r \in \mathbb{N}$  and a very ample line bundle  $L$ , such that

$$c_2 \geq \alpha(r, c_1, L),$$

where  $\alpha$  is the Li-Qin constant introduced before.

- In terms of the above definition, the Li-Qin existence Theorem has the following obvious consequence:

## COROLLARY

On a general type surface  $X$  "good of type  $(r, L)$ ", there exists  $L$ -stable omalous vector bundles of rank  $r$ .

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# EXAMPLES OF "GOOD" SURFACES I

- This last section is devoted to the illustration of the above results on a concrete class of examples:  $X_d$  a smooth surface of degree  $d$  in  $\mathbb{P}^3$ .
- Well known computations gives for  $X_d$  the following values of invariants:

$$c_2 = d^3 - 4d^2 + 6d \quad c_1^2 = d(d-4)^2$$

- Also, by Noether formula,

$$p_g = \chi(\mathcal{O}) - 1 = \frac{c_1^2 + c_2}{12} = \frac{d^3}{6} + \dots$$

- So, the leading term in  $d$  which appear in the Li-Qin constant  $\alpha$  is

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# EXAMPLES OF "GOOD" SURFACES II

- As consequence, for  $2 \leq r \leq 6$  and  $d \gg 0$  we have  $c_2 \geq \alpha$ .
- So, one can apply the Li-Qin construction, obtaining the following:

## THEOREM (-,2015)

*There is an explicitly computable constant  $d_0$ , such that for all  $d \geq d_0$  and all  $2 \leq r \leq 6$ , on any smooth surface  $X_d \subset \mathbb{P}^3$  of degree  $d$  there exists a stable omalous bundle of rank  $r$ .*

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# FURTHER DIRECTIONS

- As further directions, an open question asked by Donagi et al. in the paper "(0, 2) Quantum Cohomology" published in Proceedings of Symposia in Pure Mathematics, Vol 85, 2012, concern the computation of the quantum sheaf cohomology for other sheafs than the tangent bundle.
- Of course, the above constructed stable omalous bundles are good candidates for such a computation.
- Moreover, due to the range of their ranks, less than 7, this question is very interesting viewing the following:

## CONJECTURE: GUFFIN, 2011

For omalous bundles  $E$  of rank  $r \leq 7$  on a smooth variety, the quantum sheaf cohomology  $QH^*(X, E)$  exists.

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THANK YOU!